Inclusive polarised $J/\psi$ production in high energy hadron collisions

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Degree Project in Physics for Engineers

"ALICE is also working hard to help solve other puzzles in $J/\psi$ production proton-proton collisions, in particular by studying the degree of polarization. A first result shows that the $J/\psi$ are essentially unpolarized. Theorists are now working to establish if such behaviour is compatible with the NRQCD approach." From - ALICE unveils mysteries of the $J/\psi$ - CERN Courier, March 2012.
Abstract

We study the inclusive polarised and unpolarised productions of $J/\psi$ coming from $\chi_c$ radiative decays in high energy hadron-hadron collisions. It is carried out at leading order in the Non-Relativistic QCD approach combined with the $k_\perp$-factorisation framework. The polarised and unpolarised observables are analysed in detail both qualitatively and quantitatively for various polarisation frames and energies.

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I. INTRODUCTION

For a few decades by now, the $J/\psi$ meson, one of the bound states of charm/anti-charm quarks (called charmonia), remains one of the most important sources of information about both perturbative and non-perturbative Quantum Chromodynamics (QCD) interactions of heavy quarks and gluons (together called parton). It is the subject of intensive ongoing theoretical and experimental studies in the high energy particle physics. Its production rates measurements give rise to a deeper understanding of the underlined QCD production mechanisms at very small parton momentum fractions and relatively small scales. Inclusive observables, like the production cross sections and various polarisation characteristics, give a chance to uncover long-standing issues of strong interactions in different kinematical regimes related to an interplay between hard and soft QCD physics.

The $J/\psi$ meson is one of the well-studied charmonium states, both experimentally and theoretically. From a Quantum Mechanics point of view it can be considered as a quantum harmonic oscillator: it is the $1^3S_1$ state, denoted as $n^{2S+1}L_J$, with the following set of intrinsic quantum numbers: the main quantum number is $n = 1$, the intrinsic angular momentum is $S = 1$, the orbital angular momentum is $L = 0$, and, consequently, its total angular momentum is $J = 1$ (such that $|L - S| \leq J \leq L + S$). Its possible polarisation states correspond to $\lambda = -1, 0, +1$ (such that $-J \leq \lambda \leq J$).

Recent theoretical developments of the heavy quarkonia production have followed from the application of the so-called Non-Relativistic QCD (NRQCD) [1]. It is an effective theory that disentangles physics at the scale of the heavy quark of mass $m_q$, relevant to the production of a heavy quark pair, from physics at the scale given by the bound state’s binding energy $m_q v^2$, relevant to the formation of the quarkonium ($v$ is the relative quark-antiquark velocity). The inclusive production cross section in high-energy proton-antiproton collisions (at the center-of-mass energy $\sqrt{s} = 1.8$ TeV at Tevatron) for the unpolarised $J/\psi$ meson in the NRQCD approach combine with the so-called $k_t$-factorisation QCD framework has already been computed in Ref. [2] and the Tevatron experimental data [3] have been successfully described. The inclusive quarkonium production is now viewed as a two-step process, where the production of a heavy quark pair $Q\bar{Q}[n]$ in a certain angular momentum and color state $n$ (for the color singlet $J/\psi$ state $n = 3S_1^{(1)}$ contributing at the lowers order in the non-relativistic velocity expansion in relative quark-antiquark velocity $v$) is described by a perturbatively calculable short-distance cross section $d\sigma_n$ and the subsequent quarkonium formation is parameterized by a non-perturbative matrix element to be determined from the data [4].

The self-consistent experimental and theoretical description of the $J/\psi$ polarisation at hadron colliders represents one of the most difficult challenges faced by heavy quarkonium production models (see, for example, Refs. [6, 7] and references therein). The first major issue is the existing strong disagreement between experimental data and theory. Besides that, the experimental knowledge itself seems to be contradictory when different polarisation measurements are compared together (for more details, see Ref. [8]). A resolution of such issues requires a deeper understanding of the underlined quarkonia production mechanisms from the theoretical QCD point of view.

---

1 The inclusive production means that the initial colliding hadrons may be destroyed during the scattering process and create a multitude of extra hadronic products that are not measured.
The current Master project aims at computing inclusive $J/\psi$ prompt (gluon-initiated) production cross section for each of its polarisation states in high-energy proton-(anti)proton collisions as well as comparing the results of different theoretical models with the latest experimental data both on unpolarised and polarised observables (specifically, at the Tevatron and the Large Hadron Collider (LHC) at CERN). Basic concepts of polarised quarkonia production and major experimental issues are discussed in detail e.g. in Ref. [5] and in references therein. In this work, important issues of polarised $\chi_c$ and $J/\psi$ production at hadron-hadron colliders such as the relation between the parameters of the angular distributions and the angular momentum composition of heavy quarkonia, the dependence of the polarisation observables on the reference frames, the interplay between observed decay and production kinematics, and the influence of detector acceptances on the comparison between experimental data and theoretical calculations are analysed in detail both qualitatively and quantitatively.

A. $J/\psi$ production mechanism

One of the basic principles of QCD as a theory of strong interactions is the factorisation of short-distance and long-distance interactions. Concerning the heavy quarkonia production, in general, (and $J/\psi$ meson production, in particular) such a QCD factorisation concept is realized both in the initial and final states. The initial-state factorisation separates the large-distance parton density functions (PDFs)\(^2\) in the initial-state hadrons (the soft part) and the short-distances hard matrix element of parton rescattering (the hard part), e.g. $gg \rightarrow Q\bar{Q}$, where $Q$ and $\bar{Q}$ are heavy ($m_Q \gg \Lambda_{QCD}$) quark and antiquark, respectively. The final-state factorisation, in turn, assumes that the parton production/scattering process is independent of the hadronisation process of the produced $Q\bar{Q}$-pair into a particular quarkonium state, and allows one to employ ideas of the NRQCD framework.

There are two major experimentally distinguishable ways to produce $J/\psi$ in high energy hadron collisions: the non-prompt production going indirectly through the decays of produced $B$-mesons, and the prompt production which goes either directly to $J/\psi$ from a charm/anti-charm quark pair produced in a gluon-gluon fusion subprocess, and indirectly – through decays of parent charmed particles (e.g. other heavier charmonia $\chi_c$ or $\psi'$). In the current thesis, we limit our study to the latter case involving only charm quarks at all stages of the $J/\psi$ production process.

In order to produce a charmonium state promptly, we first need to produce the pair of charm $c\bar{c}$ quarks in a hard interaction. In high energy hadron-hadron collisions, the $c\bar{c}$ quarks are mainly produced in the interaction and/or fragmentation of energetic gluons emitted by colliding hadrons. The $c\bar{c}$ pair can then form a bound state either directly to the $J/\psi$ state (see fig. 1) or indirectly – by binding them first into a $\chi_{cJ}$ state ($1^3P_J$ with quantum numbers $J^{Parity} = 0^+, 1^+ \text{ or } 2^+$), which can then decay into the $J/\psi$ state by radiating a photon (see fig. 2), or into $\psi'$ (the $2^3S_1$ state) decaying into $J/\psi$ and a set of pions. The indirect channel going through the decay of $\chi_c$ states gives a significant contribution to the $J/\psi$ production rate, and will be studied below in detail. We postpone the analysis of the direct $g^*g^* \rightarrow J/\psi$

\(^2\) This is essentially the probability density to find a parton carrying a fraction $x$ of the initial momentum of the parent hadron.
production channel for a later study.

![Diagram of QCD mechanism for $J/\psi$ production](image1.png)

**FIG. 1:** The QCD mechanism leading to the inclusive *direct* $J/\psi$ production in a hadron-hadron collision.

![Diagram of QCD mechanism for $J/\psi$ production](image2.png)

**FIG. 2:** The QCD mechanism leading to the inclusive *indirect* $J/\psi$ production in a hadron-hadron collision going through the intermediate decaying $\chi_c$ state.

There are, of course, other ways to produce the $J/\psi$ meson but they are either negligible or not interesting in our consideration of polarised charmonia production. For example, by replacing both gluons by two fusing photons, we obtain the electromagnetic (e.m.) production mechanism. It can be easily demonstrated, that such a contribution is strongly suppressed compared to the QCD mechanism at small longitudinal momentum fractions $x_{1,2}$ carried by the initial-state gluons (e.g. at Tevatron and LHC energies). Indeed, the e.m. interaction constant $\alpha_{em}$ is much smaller than the strong one, and the probability to find photons in a hadron, given by the e.m. form factor of an interacting hadron, is much smaller than the probability to find gluons with energy sufficient to produce a bound charmonium state. Consequently, the cross section of this e.m. mechanism turns out to be much smaller as compared to the QCD one.

Another way to produce $J/\psi$, mentioned above, is through the intermediate production of the $\psi'$ charmonium state ($2^3S_1$), instead of $\chi_cJ$, which decays then into $J/\psi$ and a set of pions. But as pions have spin equal to zero, they cannot be polarised and as a consequence, the polarisation of $J/\psi$ will be the same as the polarisation of the $\psi'$ meson, so in this mechanism the produced $J/\psi$ polarisation is the same as the one in the direct $J/\psi$ production.
As was already emphasized above, in high energy hadron collisions, the c and \( \bar{c} \) quarks are mainly produced from the hard interaction of two gluons (i.e. at the hard scale \( \mu \) typically given by their invariant mass). This is because the gluon distribution function (gluon PDF) of the longitudinal momentum fraction \( x \) and the factorisation scale \( \mu, xg(x, \mu^2) \), strongly grows at small \( x \), and completely dominates the production cross section at high energies \( \sqrt{s} \). The factorisation scale \( \mu \) in this sense represents the boundary between hard QCD matrix element \( g^*g^* \rightarrow c\bar{c} \) (at momentum scales larger than \( \mu \)) and the soft interactions (at momentum scale smaller than \( \mu \)) absorbed into the universal gluon PDF. The first step is, therefore, to study the soft gluon QCD dynamics and factorisation mechanisms through the use of PDFs (or unintegrated PDFs) and their perturbative evolution.

The second step is to compute the hard matrix element of the \( c\bar{c} \) quark pair production in a fusion of virtual gluons, by using the Feynman diagram technique at the leading order in strong coupling constant \( \alpha_s \). Subsequently, the produced \( c\bar{c} \) pair should be hadronised into a particular charmonium state. This step is achieved by employing the NRQCD technique [1, 9] of projection of a heavy quark-antiquark pair, with a small relative velocity of charm quarks compared to their mass \( v \ll m_c \simeq 1.4 \text{ GeV} \), onto a particular bound state; in our case – the \( \chi_{cJ} \) state.

The third step, to be done only in the indirect production case, is the radiative decay of \( \chi_{cJ} \) into the \( J/\psi \) meson and a photon. This study includes the angular distribution of the \( J/\psi \) meson, the branching ratios of incident particles and the links between \( \chi_{cJ} \) and \( J/\psi \) polarisation observables, as well as a comparison with available experimental data from Tevatron and LHC.

The last step is technically similar to the third one. It deals with decay of the \( J/\psi \) meson into a lepton pair. From the experimental point of view, the decay channel into \( \mu^+\mu^- \) pair is the most convenient and well-studied one, so we will be primarily focused on it.

B. Background and formalism

In this Section, we will review the QCD theoretical background relevant for our studies, and make an outline of basic methods used in explicit analytical calculations.

1. Color Singlet Model vs Color Octet Model

Originally, the inclusive heavy quarkonium production was described within the color singlet model in the so-called collinear QCD factorisation approach [9].

Due to confinement of color, in a real experimental situation the color quantum number, and hence colored partons (gluons and quarks), can never be observed at large distances in a detector; partons turn out to be absolutely confined inside hadrons and cannot move at a macroscopic distance larger than the hadronic size \( \sim 1 \text{ fm} \). The production of a color singlet ("colorless") charmonium state requires a color singlet \( c\bar{c} \) pair to start with, which means \( i = \bar{k} \), where \( i \) and \( k \) are the triplet color indices of charm quarks. In the framework of the color singlet model (CSM) [26, 27], the \( c\bar{c} \) pair is directly produced in a color singlet state (see fig. 3), ready for hadronisation into a bound state.

In opposite to the CSM, in the framework of the color octet model (COM) [28, 29] the \( c\bar{c} \) pair is not directly produced in the color singlet state, but is produced in a color octet state with color quantum number of a gluon \( (i \neq \bar{k}) \) and then is turned into the color singlet state.
by soft gluon exchanges with hadron remnants at large distances compared to the scale of the hard production subprocess (see fig. 4). Such a screening of the color charge may be very important for analysis of observables (see e.g. Ref. [30] and references therein).

2. Collinear vs transverse momentum dependent (or $k_\perp$) factorisation in QCD

The QCD theory gives quantitative predictions about the logarithmic rate of change of the gluon PDF when the momentum scale $\mu$ is varied [12, 14]. The commonly used procedure is based, first, on parameterizing the dependence of the PDF on the variable $x$ at some low value of $\mu^2 = \mu_0^2 \sim 1 \text{ GeV}^2$, which at the same time is large enough such that unknown terms in the perturbative Renormalisation Group evolution equations can be neglected, and then evolving these “input PDFs” up to the hard scale $\mu^2$ through an evolution provided by the so-called DGLAP [17], BFKL [18] or CCFM [19] evolution equations.

As illustrated in fig. 5, the most important evolution contribution at small $x$ is the gluon bremsstrahlung, with typical behavior, as in Quantum Electrodynamics [10], of strongly growing in the infrared (soft) and/or collinear region (soft and collinear divergences). The QCD factorisation provides the universal and general tool for isolating the infrared divergences, resuming them at a particular order of Perturbation Theory, and absorbing them into an unmeasurable bare PDF, which is thereby transformed into the physical measurable PDF and can be constrained from an experiment. Such a new object is proven to be universal [13], such that it can be extracted in one process and used for description of another. As a pay-off for the cancelation of divergences, the physical PDF acquires a logarithmic evolution w.r.t. a new scale at which the separation of divergences has been made
FIG. 5: The illustration of the perturbative gluon DGLAP evolution through the resummation of
the gluon bremsstrahlung contributions strongly ordered in transverse momenta (or, equivalently,
in intermediate gluon virtualities).

– the factorisation scale $\mu$. Among the several different resummation techniques (governed
by the QCD evolution equations) of this perturbative that have been developed so far, we
can mention: the DGLAP (Dokshitzer, Gribov, Lipatov, Altarelli, Parisi) approach [17],
also known as the collinear factorisation technique, which resums the collinear (i.e. parton
transverse momentum $p_T \to 0$) divergent contributions $\sim [\alpha_s \ln p_T]^{n}$; the BFKL (Balitsky,
Fadin, Kuraev, Lipatov) approach [18], also known as the transverse momentum dependent
or $k_\perp$-factorisation, which resums the infrared (i.e. parton momentum $p \to 0$ or equivalently
$x \to 0$) contributions $\sim [\alpha_s \ln(1/x)]^{n}$, and the CCFM (Catani, Ciafaloni, Fiorani, March-
esini) approach [19] that attempts to cover both collinear and infrared regions by considering
the color coherence effects (for a review on these topics, see e.g. Refs. [14] and references
therein).

The standard collinear factorisation approach [13] includes two major steps: the fac-
torisation of the inclusive production cross section into the integrated PDFs and the hard
subprocess cross section, and the hard matrix element calculation neglecting the gluon vir-
tualities (or transverse momenta) as compared to the hard scale $\mu$.

According to the collinear factorisation approach [13], the inclusive charmonium
($((Q\bar{Q})_J \equiv \chi_{cJ}$ or other quarkonium state) production cross section can be written in the
following way schematically sketched in fig. 6 [31]:

$$
\frac{d\sigma(pp \to X(Q\bar{Q})_J)}{dy} \propto \int_{x_1}^{1} d\xi_1 g(\xi_1, \mu^2) \int_{x_2}^{1} d\xi_2 g(\xi_2, \mu^2) \cdot \frac{d\hat{\sigma}(gg \to (Q\bar{Q})_J)}{dy}
$$

where $\xi_{1,2}$ are the longitudinal momentum (the part of the momentum along the beam axis)
fractions taken by the gluons from the colliding hadrons, $x_{1,2} = (M_{Q\bar{Q}}/\sqrt{s}) \exp(\pm y)$ are here
the minimum longitudinal momentum fractions required for the meson production (they are
written in terms of the meson rapidity, $y$, $\mu$ is the chosen factorisation scale for the hard reaction (usually of the order of the charmonium mass), $g(\xi_1, \mu^2)$ is the gluon PDF, and $\hat{\sigma}(g^* g^* \to (Q\bar{Q})_J)$ is the hard subprocess cross section including NRQCD projection of the $c\bar{c}$ pair onto a particular bound charmonium state. The star in $g^*$ means that the gluons are taken to be virtual, so the exact kinematics in the matrix element is implied. The product $g(\xi, \mu^2) d\xi$ provides the probability to find gluons with a certain momentum fraction in the interval $[\xi, ..., \xi + d\xi]$ at a given scale $\mu$ in the colliding hadrons [14].

Motivated by the longitudinal motion of the hadrons, in the collinear approach the gluon transverse momenta are taken to be zero or negligibly small in the hard matrix element as compared to the hard factorisation scale $\mu$, i.e. $k_{1,2}^2 \approx k_{1,2}^2 / \mu^2$. As $k_{1,2}^2 \approx 0$, the gluons are treated to be on-shell in the hard production matrix element, which is problematic for the production of e.g. the axial-vector $\chi_{c,J}=1$ state. Indeed, as the two on-shell gluons have a polarisation $\lambda$ equal to $\pm 1$, the Landau-Yang theorem forbids their productions [16] (see also Ref. [21]): the possible $J$ that can be produced are the absolute values of the combinations of the gluon polarisations, i.e. with $\lambda = \pm 1$ the possible $J$ are 0 and 2 only. So it is essential that the $J=1$ $\chi_c$ state can only be produced by a fusion of off-shell gluons.

The predictions for the differential (in meson transverse momentum) unpolarised cross section in the Next-to-Leading Order (NLO) collinear factorisation and with the color-singlet contribution only, underestimates the Tevatron experimental data by a factor 50-100 (see fig. 6 in Ref. [9]). As the fusing gluons have no transverse momentum, the charmonium transverse momentum originates from a non-longitudinal gluon emission somewhere in the process. The proposed solution for this discrepancy was too add the COM contribution that turned out to dominate the total cross section: the sum of the CSM and COM contributions is in good agreement with the data (see fig. 6 in Ref. [9]). However, the COM introduces in calculable non-perturbative parameters, the color octet matrix elements, which are determined by a fit to the data. Moreover, it does not give a good description of polarised charmonium production data (see e.g. fig. 1 in Ref. [32]).

\[ \text{The rapidity of a particle quantifies the boost along the beam axis required to go from the laboratory frame to the frame where the particle has only a transverse momentum. It is computed from the particle’s energy and longitudinal momentum with } y = \frac{1}{2} \ln \frac{E + p_T}{E - p_L}. \]
A new approach was proposed: the use of the transverse momentum dependent or $k_\perp$-factorisation instead of the collinear one described above (for more details on this technique, see Refs. [15]). The $k_\perp$-factorisation technique includes the factorisation of the cross section (see Eq. (1.1)) with the gauge-invariant off-shell matrix element, where the gluon transverse momenta (or virtualities) are not taken to be equal to zero, and with explicitly transverse gluon polarisation vectors (see Eq. (3.12) below), and the so-called unintegrated gluon distribution functions (or UGDFs) $F(x,k_t^2)$ related with the standard integrated ones as

$$xg(x,\mu^2) = \int_{\mu^2}^{\infty} \frac{dk_t^2}{k_t^2} F(x,k_t^2,\mu^2)$$  \hspace{1cm} (1.1)

This technique, even at the Leading Order (LO) of Perturbation Theory, is known to give excellent results in the unpolarised case (see fig. 5 in Ref. [2]) with the CSM contribution only, which turns out to dominate the cross section in opposite to the one obtained in the collinear factorisation approach. The $k_\perp$-factorisation also enables production of the axial-vector charmonium state $\chi_c(1^{+})$, production of which is forbidden by the Yang-Landau theorem in the collinear factorisation framework: the gluons are now off-shell with virtualities given by their transverse momenta squared $k_{1,2}^2 \simeq k_{1,2}^2/t$, so the amplitude $g^*g^* \to \chi_c,J=1$ does not vanish. In the following calculations we apply the $k_\perp$-factorisation technique and derive exact (off-shell) hard matrix elements for all $J = 0, 1, 2$ angular momentum (and respective polarisation) states of $\chi_c$, with their subsequent decay to polarised $J/\psi$ states.

C. Motivation for the study

The $J/\psi$ polarisation measurements can serve as a crucial test of a QCD theoretical model for the heavy quarkonia production: as the couplings between particles are spin-dependent, the data may support or disprove existence of specific interactions as well as to constrain the theoretical uncertainties from specific production mechanisms. Therefore, the study of the polarised charmonia production might allow us, first, to conclude the controversy between the collinear and $k_\perp$-factorisation approaches, on the one hand, and to determine whether the COM is the dominating production process in this reaction or not, on the other hand. It is very likely that the $J/\psi$ meson is, at least, partially polarised as recent experimental data have suggested, and that the polarisation changes as a function of the c.m.s. energy, due to the onset of different production mechanisms: the LO processes at low energy with increasingly important higher order contributions at higher energies. It is, thus, advisable to perform measurements and theoretical predictions at different energies and kinematical ranges.

Even more importantly, as the study of the polarised charmonia production is much more sensitive to details of the QCD production mechanism than in the unpolarised case, we expect to gain a better understanding of the underlying QCD dynamics encoded in the soft PDFs. Indeed, as the unpolarised production is the sum of the polarised contributions, we can, in principle, obtain a reasonable unpolarised prediction since large uncertainties for each polarisation can largely compensate each other when summed altogether.

The quest for the correct QCD mechanisms through the polarised charmonia production may, therefore, allow us to give a better insight in the objects that participate to the reaction and ripen the description of the corresponding production mechanisms: the structure of colliding hadrons at long distances, the reaction between incident partons at short distances,
the gluon virtualities that allow the production of $J = 1$ states, the charmonium production from the $c\bar{c}$ pair (i.e. the hadronisation into small-mass systems) and the charmonia decay process.

Additionally, a better understanding of charmonium polarised production in unpolarised hadron-hadron collisions in our case can be an important basis for the future studies of the $J/\psi$ production in polarised hadron-hadron collisions (e.g. at BNL RHIC), as well as in $pA$ and $AA$ collisions at the LHC probing properties and dynamics of the quark-gluon plasma.
II. GENERAL KINEMATICS AND THE CHARMONIA PRODUCTION CROSS SECTIONS

A. Kinematics of the inclusive charmonium production

The kinematics for the inclusive charmonium production are shown in figure 7.

![Diagram of inclusive charmonium production](image)

FIG. 7: General kinematics of inclusive charmonium production.

The particles involved in charmonium production and their variables and parameters are summed up in the table I.

<table>
<thead>
<tr>
<th>4-momentum</th>
<th>rest mass</th>
<th>color</th>
<th>others</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hadron 1</td>
<td>$h_1$</td>
<td>negligible</td>
<td></td>
</tr>
<tr>
<td>Hadron 2</td>
<td>$h_2$</td>
<td>negligible</td>
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<tr>
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</tr>
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<td>$m$</td>
<td>$i$</td>
</tr>
<tr>
<td>Quark 2</td>
<td>$p_2$</td>
<td>$m$</td>
<td>$k$</td>
</tr>
<tr>
<td>Charmonium</td>
<td>$P$</td>
<td>$M$</td>
<td>Helicity $\lambda$, Quantum numbers: $J, L$ and spin $S$</td>
</tr>
</tbody>
</table>

TABLE I: The particles involved in charmonium production and their kinematical variables/parameters.

First, the Dirac notation $\gamma^\sigma p^\sigma$, with any 4-momentum $p$, will be used for the matrix product $\gamma^\sigma p^\sigma$. We will also use the notations for light-cone components $p^+ \equiv p^0 + p^3$ and $p^- \equiv p^0 - p^3$.

In the center-of-mass system (c.m.s.) frame the hadron momenta can be written as:

$$h_1 = \frac{\sqrt{S}}{2} (1, 0, 0, 1), \quad h_2 = \frac{\sqrt{S}}{2} (1, 0, 0, -1)$$

(2.1)
where the Mandelstam variable $s$ is defined by $s \equiv (h_1 + h_2)^2 = 2(h_1 h_2) = 4E_{c.m.s.}^2$ as $h_{1,2}^2 = m_{\text{hadron}}^2 \sim 1 \text{ GeV}^2$ are negligible compared to hadron energies in typical high energy collisions. The factor $\sqrt{s}/2$ in (2.1) is obtained with $\sqrt{s} = h_1^0 + h_2^0$ and $h_1 = -h_2$ (a bold faced character means 3-vector) in the c.m.s. frame.

The decompositions of gluon momenta into longitudinal and transverse components are given by:

$$k_1 = k_{1,L} + k_{1,t} = x_1 h_1 + k_{1,t}, \quad k_2 = k_{2,L} + k_{2,t} = x_2 h_2 + k_{2,t}, \quad 0 < x_{1,2} < 1 \quad (2.2)$$

where $x_{1,2}$ are the longitudinal momentum fractions taken by the virtual gluons to the hadrons. They are defined by:

$$x_1 = \frac{k_{1,t}^+}{h_1^+}, \quad x_2 = \frac{k_{2,t}^+}{h_2^+} \quad (2.3)$$

From the typical values in high energy hadron-hadron collisions at Tevatron ($\sqrt{s} = 1.8 \text{ TeV}$): $x_{1,2}^2 \sim 10^{-4} - 10^{-6}$, $h_{1,2}^2 = m_{\text{hadron}}^2 \sim 1 \text{ GeV}^2$ and $k_{1,2,t}^2 \sim 1 \text{ GeV}^2$ within $k_\perp$ factorisation (whereas in collinear factorisation $k_{1,2,t}^2 \sim 0$), we deduce:

$$x_1^2 h_1^2 \ll k_{1,t}^2, \quad x_2^2 h_2^2 \ll k_{2,t}^2 \quad (2.4)$$

and with (2.2) it yields:

$$k_1^2 = k_{1,t}^2, \quad k_2^2 = k_{2,t}^2 \quad (2.5)$$

The relation (2.5) relates gluon virtualities to gluon transverse momenta squared.

The charmonium momentum is $P = k_1 + k_2 = p_1 + p_2$ by momentum conservation, therefore:

$$M^2 = P^2 = (k_1 + k_2)^2 = (p_1 + p_2)^2 \quad (2.6)$$

which gives with (2.5):

$$2k_1 k_2 = M^2 - k_{1,t}^2 - k_{2,t}^2 \quad (2.7)$$

Defining $P_t^2 = (k_{1,t} + k_{2,t})^2$ and using (2.4) we obtain:

$$M_T^2 \equiv M^2 - P_t^2 = (k_{1,L} + k_{2,L})^2 + (k_{1,t} + k_{2,t})^2 - (k_{1,t} + k_{2,t})^2$$

$$= (x_1 h_1 + x_2 h_2)^2$$

$$= 2x_1 x_2 h_1 h_2$$

$$= sx_1 x_2 \quad (2.8)$$

By neglecting the quark binding energy we can write the following approximation for the charmonium mass $M$:

$$M \simeq 2m \quad (2.9)$$

Defining the relative momentum of the produced quark and antiquark $q = (p_1 - p_2)/2$ and using (2.6) and (2.9) we get:

$$q^2 \ll M^2 \quad (2.10)$$
B. Inclusive charmonium production cross section

Within the framework of the $k_\perp$ factorisation approach, the general cross section becomes [2]:

$$\sigma_{J,\lambda} = \frac{1}{8(2\pi)} \int \frac{d^3P}{P^0} d^2k_{1,t} d^2k_{2,t} \delta^2(k_{1,t} + k_{2,t} - P_t) \times$$

$$\times \mathcal{F}(x_1, k^2_{1,t}) \frac{1}{(k^2_{1,t})^2} \left[ \frac{V_{J,\lambda}^c V_{J,\lambda}^{c2}}{(N_c^2 - 1)^2} \right] \frac{1}{(k^2_{2,t})^2} \mathcal{F}(x_2, k^2_{2,t}) \quad (2.11)$$

where $P^0$ is the charmonium energy and the factor $(N_c^2 - 1)^2$ (where $N_c = 3$) comes from the averaging over the $(N_c^2 - 1)$ possible gluon color states for each soft gluon ($g_1$ and $g_2$). The terms $1/(k_{1,2,t})^2$ are their propagators. The unintegrated gluon distribution functions $\mathcal{F}(x_1, k^2_{1,t})$ are the ones given in (1.1). The heavy charmonium production amplitude $V_{J,\lambda}^{c2}$ (called below as vertex factor) corresponds to the projection of the hard amplitude ($g^* g^* \rightarrow c \bar{c}$) onto the singlet charmonium bound states. The explicit expression of the vertex factor will be discussed in detail in Section III.

We can introduce a more practical set of variables to describe the phase space, as for example $(y, |P_t|)$ (once again a bold faced character means 3-vector) with $y$ the rapidity defined by:

$$y = \frac{1}{2} \ln \left( \frac{E + P_z}{E - P_z} \right) = \ln \left( \frac{E + P_z}{\sqrt{M^2 + P^2_t}} \right) \quad (2.12)$$

where $E^2 = M^2 + P^2_z + P^2_t$ ($E = P_0$ being the charmonium energy). From the relation (2.12) is easily derived:

$$P_z = \sqrt{M^2 + P^2_t} \sinh y, \quad E = \sqrt{M^2 + P^2_t} \cosh y \quad (2.13)$$

Then computing the Jacobian from the variables $(P_t, P_z)$ to $(P_t, y)$ and (2.13) yields:

$$d^3P = dP_z d^2P_t = \sqrt{M^2 + P^2_t} \cosh y \, dy \, d^2P_t \quad (2.14)$$

Finally with (2.14) we obtain the differential cross section with the new set of variables:

$$\frac{d^3\sigma_{J,\lambda}}{dy d^2P_t} = \frac{1}{8(2\pi)} \int d^2k_{1,t} d^2k_{2,t} \delta^2(k_{1,t} - k_{2,t} - P_t) \times$$

$$\times \mathcal{F}(x_1, k^2_{1,t}) \frac{1}{(k^2_{1,t})^2} \left[ \frac{V_{J,\lambda}^c V_{J,\lambda}^{c2}}{(N_c^2 - 1)^2} \right] \frac{1}{(k^2_{2,t})^2} \mathcal{F}(x_2, k^2_{2,t}) \quad (2.15)$$
III. THE HARD SUB-PROCESS CROSS SECTION

The directly produced charmonium state will be $\chi_{cJ}$ in the rest of the thesis: the cases of the direct $J/\psi$ and $\psi'$ will not be studied in this work. To compute the hard cross section (2.11) we need first to compute the vertex factors.

A. Projection of the hard amplitude $g^*g^* \rightarrow c\bar{c}$ onto singlet charmonium bound states

Projection of the hard amplitude onto the singlet charmonium bound state $V_{J\lambda}^{c\bar{c}2}$ is given by a 4-dimensional integral over relative momentum of the produced quark and antiquark $q = (p_1 - p_2)/2$ [20][2]:

$$V_{J\lambda}^{c\bar{c}2}(k_1, k_2) = \mathcal{P}(c\bar{c} \rightarrow \chi_{cJ}) \cdot \Psi_{ik,\mu\nu}^{c\bar{c}2}(p_1, p_2) = 2\pi \sum_{i,k} \sum_{L_z S_z} \frac{1}{\sqrt{m}} \int \frac{d^4q}{(2\pi)^4} \delta \left(q^0 - \frac{q^2}{M}\right) \times$$

$$\times \Phi_{L=1,L_z}(q) \times \langle L = 1, L_z; S = 1, S_z|J, J_z\rangle\langle 3i, 3k|1\rangle \text{Tr} \{\Psi_{ik}^{c2} \mathcal{P}_{S=1,S_z}\} \quad (3.1)$$

Some explanations about the projection equation:

* $\Psi_{ik}^{c\bar{c}2}(p_1, p_2)$ is the amplitude of the process $(gg \rightarrow c\bar{c})$ and $\mathcal{P}(c\bar{c} \rightarrow \chi_{cJ})$ is the operator that projects the $c\bar{c}$ pair onto the charmonium bound states $\chi_{cJ}$.

* $\sum_{i,k}$ is the sum over all possible colors for the intermediate $c\bar{c}$ pair. As the projection into $\chi_c$ requires a color singlet state meson we should sum only over $i = \bar{k}$. This is achieved with the $\delta^{ik}$ term included in the explicit form of the $\langle 3i, 3k|1\rangle$ terms. The latter are the Clebsch-Gordan coefficients (related to the color part of the projection amplitude) in the color space defined by the expansion of the $\chi_c$ color singlet state $|1\rangle$ in the $c\bar{c}$ color state basis: $|1\rangle = \sum_{i,k} |3i, 3k\rangle (3i, 3k|1\rangle$. The term $|3i, 3k\rangle$ is included in $\Psi_{ik}^{c\bar{c}2}$.

* $\sum_{L_z,S_z}$ is the sum over all $\chi_{cJ}$ orbital angular $L_z$ and spin $S_z$ projected momenta: $-L \leq L_z \leq L$ and $-S \leq S_z \leq S$. Moreover $J$ the total angular momentum is such as $|L - S| \leq J \leq L + S$ i.e. $J = 0, 1$ or $2$ when $S = 1$ and $L = 1$. The coefficients $\langle L = 1, L_z; S = 1, S_z|J, J_z\rangle$ are related to the angular momentum part of the projection amplitude. Indeed these terms are the Clebsch-Gordan coefficients in the total angular momentum space and are defined by the expansion of the $\chi_{cJ}$ total angular momentum states $|J, J_z\rangle$ in the $\chi_{cJ}$ spin and orbital angular momentum basis: $|J, J_z\rangle = \sum_{S_z, L_z} |S, S_z; L, L_z\rangle \langle S, S_z; L, L_z|J, J_z\rangle$. The term $|S_z, S_z; L, L_z\rangle$ is included into $\Phi_{L=1,L_z}(q)$ and $\mathcal{P}_{S=1,S_z}$.

* The integral $\int \frac{d^4q}{(2\pi)^4} \delta \left(q^0 - \frac{q^2}{M}\right) \Phi_{L=1,L_z}(q)$ is the integration of the $\chi_{cJ}$ momentum wave-function $\Phi_{L=1,L_z}$ over small relative momenta (restriction obtained by the delta function) as it is required, for the quarks to bound into a quarkonium, that their relative velocity being small compare to their combine velocity. Indeed if the relative
momentum is small [9], in one of the quark rest frame the total "relative" energy can be written as:

\[ E_{\text{relative}}^\text{total} = q^0 = E_{\text{relative}}^\text{kinetic} + E_{\text{relative}}^\text{mass} \simeq Mv^2 + (\gamma - 1)M \simeq Mv^2 \simeq q^2 / M \]

where \( v \) is the relative velocity of \( c \) and \( \bar{c} \). The approximation \( (\gamma - 1) \simeq 0 \) comes from the non relativistic quark relative velocity which implies the approximation of the related Lorentz factor to \( \gamma \simeq 1 \).

* The Tr(\( \Psi_{ik}^{c_1c_2}P_{S=1,S_z} \)) is the projection operator acting on the hard amplitude. The trace comes from the sequence of Dirac spinor indices and is computed in Section III B 1 and III B 2.

B. The projection operator acting on the hard amplitude at tree level

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig8.png}
\caption{The three Feynman diagrams (t, u and s channels) of \( gg \to c\bar{c} \) at tree level.}
\end{figure}

1. The projection operator acting on the hard amplitude of the t-channel

Only in this section we define \( \lambda_1 \) and \( \lambda_2 \) as being the hadron helicities (or polarisations). The notation \( \hat{p} \) is equivalent to the Faynman slash notation, i.e \( \hat{p} = \gamma_{\mu}p^\mu \) (where the \( \gamma \) are the usual gamma matrices of theoretical particle physics). Using the Feynman rules for the t-channel in fig. 8 one obtains the amplitude of the t-channel:

\[ i\Psi_{ij}^{c_1c_2,\lambda_1\lambda_2} = (ig_s)^2\varepsilon_{\lambda_1}^\nu(k_1)\varepsilon_{\lambda_2}^\mu(k_2) \sum_{j=1}^{N_c} t_{k_j}^{c_2} f_{j_1}^{c_1} \times \bar{u}_{\eta_1}^{S_1}(p_1)\gamma_\nu\xi(i\hat{p}_1 - \hat{k}_1 + m_{N_c}) \frac{1}{(p_1 - k_1)^2 - m_{N_c}^2} \times \]

\[ \gamma_\sigma\varepsilon_{\sigma}(p_2) \]  

\[ (3.2) \]

The meson is produced from the incoming \( c(p_1) \) and \( \bar{c}(p_2) \) corresponding to the respective Dirac spinors: \( u_{\eta_1}^{S_1}(p_1) \) and \( \bar{v}_{\eta_2}^{S_2}(p_2) \). Therefore \( P_{S=1,S_z} \) has the following expression for small relative momentum \( q \) (as the charm quark mass is important) [20]:

\[ P_{S=1,S_z} = \frac{1}{2m} \bar{v}_{\sigma}^{S_2}(p_2)\frac{\varepsilon_{\sigma}(S_z)}{\sqrt{2}} u_{\eta}^{S_1}(p_1) \]  

\[ (3.3) \]
Where \( \varepsilon^\rho(S_z) \) is the charmonium polarisation vector related to its spin.

The sequence of Dirac spinor indices in (2.1) and (2.2) gives us the trace (and therefore the liberty to make circular permutations of Dirac spinors) and summing over quark spins (unpolarised cross section) yields:

\[
\text{Tr} \left( \sum \Psi_{ij}^{c_1c_2\lambda_1\lambda_2} P_{s=1,s_z} \right) = \frac{(ig_s)^2}{2m} \varepsilon^\mu_{\lambda_1}(k_1)\varepsilon^\mu_{\lambda_2}(k_2) \frac{\varepsilon^\rho(S_z)}{\sqrt{2}} \sum_{j=1}^{N_c} \epsilon_{k_j}^c \epsilon_{j_1}^t \times \right.
\times \text{Tr} \left( \sum \tilde{u}^s(p_1)\tilde{u}^t(p_1)\gamma^\nu \left( \frac{\hat{p}_1 - \hat{k}_1 + m}{(p_1 - k_1)^2 - m^2} \right) \gamma^\mu u^s(p_2)\bar{u}^t(p_2)\gamma^\rho \right) (3.4)
\]

Using of the completeness relations for Dirac spinors and the Gell-Mann matrices relation \( \sum_{k=1}^{N_c} \epsilon_{k}^c \epsilon_j^t = \delta^{c_1c_2}/2 \) [11] we obtain:

\[
\text{Tr} \left( \sum \Psi_{ij}^{c_1c_2\lambda_1\lambda_2} P_{s=1,s_z} \right) = \frac{\delta^{c_1c_2}}{4m} (ig_s)^2 \varepsilon^\mu_{\lambda_1}(k_1)\varepsilon^\mu_{\lambda_2}(k_2) \frac{\varepsilon^\rho(S_z)}{\sqrt{2}} \times \right.
\times \text{Tr} \left( (\hat{p}_1 + m)\gamma^\nu \left( \frac{\hat{p}_1 - \hat{k}_1 + m}{(p_1 - k_1)^2 - m^2} \right) \gamma^\mu (\hat{p}_2 - m)\gamma^\rho \right) (3.5)
\]

2. The projection operator acting on the hard amplitude of the \( t \), \( u \) and \( s \) channels

As the \( s \)-channel gives only color octet quarks and anti-quarks (because they are produced from a unique color octet gluon) and as the produced charmonium must be in a color singlet state, the latter diagram does not contribute to the total amplitude. In the \( t \)-channel and \( u \)-channel the quarks and anti-quarks can be produced with complementary colors that give the color singlet meson; therefore they contribute to the total amplitude. In the same way as in Section III B 1, we obtain the total amplitude for the superposition of the \( t \)-channel, \( u \)-channel and \( s \)-channel :

\[
\text{Tr} \left( \sum \Psi_{ij}^{c_1c_2\lambda_1\lambda_2} P_{s=1,s_z} \right) = -g_s^2 \frac{\delta^{c_1c_2}}{4m} \varepsilon^\mu_{\lambda_1}(k_1)\varepsilon^\mu_{\lambda_2}(k_2) \frac{\varepsilon^\rho(S_z)}{\sqrt{2}} \text{Tr} \left[ (\hat{p}_1 + m) \times \right.
\times \left\{ \gamma^\nu \left( \frac{\hat{p}_1 - \hat{k}_1 + m}{(p_1 - k_1)^2 - m^2} \right) \gamma^\mu + \gamma^\mu \left( \frac{\hat{k}_1 - \hat{p}_2 + m}{(k_1 - p_2)^2 - m^2} \right) \gamma^\nu \right\} (\hat{p}_2 - m)\gamma^\rho \left. \right]\] (3.6)
\]

C. The amplitude \( V_{J,J'}^{c_1c_2} \) for \( \chi_c^J \) production with \( J = 0, 1 \) and 2

By combining equations (3.6) and (3.1), the vertex factor becomes:
\[ V_{\ell,\lambda}^{c_1c_2}(q_1, q_2) = -\frac{\delta^{c_1c_2}}{2m} \frac{g_5^2}{\sqrt{2m}} \varepsilon'_{\lambda}(k_1)\varepsilon'_\lambda(k_2) \times \sum_{i,k} \langle 3i, 3k|1 \rangle \times \int \frac{d^4 q}{(2\pi)^4} \delta \left( q^0 - \frac{q^2}{M} \right) \times \] 
\[ \times \Phi_{L=1,L_\pi}(q) \times \sum_{L_z,S_z} \langle L = 1, L_\pi; S = 1, S_z|J, J_\pi \rangle \varepsilon^\rho(S_z) \times \text{Tr} \left[ (\hat{p}_1 + m) \times \right] \] 
\[ \times \left\{ \gamma^\nu \left( \frac{\hat{p}_1 - \hat{k}_1 + m}{(p_1 - k_1)^2 - m^2} \right) \gamma^\mu + \gamma^\mu \left( \frac{\hat{k}_1 - \hat{p}_2 + m}{(k_1 - p_2)^2 - m^2} \right) \gamma^\nu \right\} (\hat{p}_2 - m) \gamma^\rho \right\} \] 
\[ (3.7) \]

Since \( \chi_{cJ} \) is \( L=1 \) states, it is described by a P-wave radial wave function which vanishes linearly at the origin (with a derivative \( \mathcal{R}'(0) \)). We may therefore expand the trace in (3.7) in a Taylor series around \( q=0 \), and only keep the linear terms in \( q^\sigma \) with \( \sigma = \mu, \nu, \eta \) (\( \eta \) is an arbitrary chosen index introduced for the indices to be explicit in the \( k_{1,2}, q \) scalar products). Using the general kinematic relations of Section II A we can compute the trace of expression (3.7) with Mathematica and keeping only the linear terms in \( q^\sigma \) yields:

\[
\text{Tr} \left[ \ldots \right] = \frac{8q^\sigma (g^{\mu\nu}M^2 - 2k_1^\mu k_2^\nu - 2k_1^\nu k_2^\mu - g^{\mu\nu}k_{1,t}^2 + g^{\mu\nu}k_{2,t}^2)}{M^2 - k_{1,t}^2 - k_{2,t}^2} - \frac{8q^\mu (M^2 + k_{1,t}^2 + k_{2,t}^2) (g^{\rho\sigma}M^2 - 2k_1^\rho k_1^\sigma - 2k_2^\rho k_2^\sigma + g^{\rho\sigma}k_{1,t}^2 - g^{\rho\sigma}k_{2,t}^2)}{M^2 - k_{1,t}^2 - k_{2,t}^2} + \frac{8k_2 M^2 (2k_1^\mu k_1^\nu k_1^\rho + 2k_1^\nu k_1^\rho k_1^\mu - 2k_2^\mu k_2^\nu k_2^\rho - k_1^\rho M^2)}{(M^2 - k_{1,t}^2 - k_{2,t}^2)^2} - \text{other linear terms in } q^\sigma...
\]

The integration over small relative momentum \( q \) in (3.7) gives now [2]:
\[
\int \frac{d^4 q}{(2\pi)^4} q^\sigma \Phi_{L=1,L_\pi}(q) = -\frac{i}{2\pi} \sqrt{\frac{3}{4\pi}} \varepsilon^\sigma(L_z)\mathcal{R}'(0),
\] 
(3.8)
where \( \varepsilon^\sigma(L_z) \) is the charmonium polarisation vector related to its orbital momentum.

Then using the Clebsch-Gordan identities for \( \chi_{cJ}^J \) [27]:

For \( J = 0 \):
\[
\sum_{L_z, S_z} \langle 1, L_z; 1, S_z|0, 0 \rangle \varepsilon^\sigma(L_z)\varepsilon^\rho(S_z) = \sqrt{\frac{1}{3}} \left( g^{\sigma\rho} - \frac{P^\sigma P^\rho}{M^2} \right).
\] 
(3.9)

For \( J = 1 \):
\[
\sum_{L_z, S_z} \langle 1, L_z; 1, S_z|1, J_z \rangle \varepsilon^\sigma(L_z)\varepsilon^\rho(S_z) = -\frac{i}{\sqrt{2M}} \varepsilon^{\rho\alpha\beta} P_\alpha \varepsilon_\beta(J_z).
\] 
(3.10)
where \( \varepsilon^{\rho \sigma \alpha} \) is the Levi-Civita symbol and \( \varepsilon_\beta(J_z) \) the charmonium polarisation vector related to its total angular momentum. For \( J = 2 \):

\[
\sum_{L_z, S_z} \langle 1, L_z; 1, S_z | 2, J_z \rangle \varepsilon^\sigma(L_z) \varepsilon^\rho(S_z) = \varepsilon^\sigma(\rho J_z).
\] (3.11)

Where \( \varepsilon^\sigma(\rho J_z) \) is the charmonium polarisation tensor.

In \( k_\perp \) factorisation the gluon polarisation vectors are related to their transverse momentum by:

\[
\varepsilon^\nu_{\lambda 1}(k_1) \rightarrow \frac{2k^\nu_{1,t}}{x_1 \sqrt{s}}, \quad \varepsilon^\mu_{\lambda 2}(k_2) \rightarrow \frac{2k^\mu_{2,t}}{x_2 \sqrt{s}}
\] (3.12)

The sum over all possible colors for the intermediate \( c\bar{c} \) pair, and averaged, at the amplitude element squared level, over the number \( (N_c = 3 \text{ in the fundamental representation of SU}(3)) \) of possible colors of the singlet \( c\bar{c} \) (in CSM) is given by:

\[
\sum_{i, k} \langle 3i, \bar{3}k | 1 \rangle = \frac{1}{\sqrt{N_c}}
\] (3.13)

as we are for now at the amplitude element level.

1. The amplitude \( V_{c_1c_2}^{J=0, \lambda} \)

Using the general kinematics relations and inserting (3.8), (3.8), (3.9), (3.12) and (3.13) in (3.7) we obtain with Mathematica the final vertex factor for \( J = 0 \):

\[
V_{J=0, \lambda}^{c_1c_2} = 8ig_s^2 \delta^{c_1c_2} \frac{\mathcal{R}'(0)}{M} \frac{M_t^2(2k_{1,t}^2k_{2,t}^2 - 3M_t^2(k_{1,t} \cdot k_{2,t}) + (k_{1,t} \cdot k_{2,t})(k_{1,t}^2 + k_{2,t}^2))}{(M^2 - k_{1,t}^2 - k_{2,t}^2)^2(M^2 + P_t^2)}
\] (3.14)

A similar result has been obtained in [20] where is used the approximation \( P_t \simeq 0 \).

Inspecting relation (3.14) we notice that if \( k_{1,t} \rightarrow 0 \) or \( k_{2,t} \rightarrow 0 \) then \( V_{c_1c_2}^{J=1} \rightarrow 0 \). Which means that gluon transverse momenta are necessary to get a non zero cross section.

2. The amplitude \( V_{J=1, \lambda}^{c_1c_2} \)

Inserting (3.8), (3.8), (3.10), (3.12) and (3.13) in (3.7) we obtain with Mathematica the vertex factor for \( J = 1 \):

\[
V_{J=1, \lambda}^{c_1c_2} = 2g_s^2 \frac{\mathcal{R}'(0)(c_1c_2)}{M} \frac{\sqrt{6}}{\pi M N_c} \varepsilon^\sigma(\rho J_z) \left[ k_{1,t}^\sigma k_{2,t}^\rho (x_1 h_1^\alpha - x_2 h_2^\alpha) \times \right.
\]
\[
\times (k_{1,t}^2 + k_{2,t}^2) - \frac{2}{s} h_1^\rho h_2^\sigma \left( k_{1,t}^\alpha [2k_{2,t}^2(k_{1,t} \cdot k_2) - (k_{1,t} \cdot k_{2,t})(k_{1,t}^2 + k_{2,t}^2)] - 
\]
\[
- k_{2,t}^\alpha [2k_{1,t}^2(k_{1,t} \cdot k_2) - (k_{1,t} \cdot k_{2,t})(k_{1,t}^2 + k_{2,t}^2)] \right].
\] (3.15)
Once again this result is similar to the one in [20] where the approximation $P_t \simeq 0$ is used. In expression (3.15) the indices still appear; to obtain the final expression for the vertex factor we need to fix a specific frame (in order to determine 4-vectors explicitly).

In the four dimensions 4-vector basis $(n_0^\beta, n_1^\beta, n_2^\beta, n_3^\beta)$, the charmonium polarisation 4-vector $\varepsilon_\beta(J_z)$ with a given helicity (or polarisation) $\lambda \equiv m_z = 0, \pm 1$ is [21]:

$$\varepsilon_\beta(P, \lambda) = (1 - |\lambda|)n_3^\beta - \frac{1}{\sqrt{2}}(\lambda n_1^\beta + i|\lambda|n_2^\beta)$$ (3.16)

$\lambda = 0$ corresponds to a linear polarisation along $P$, $\lambda = \pm 1$ correspond respectively to a left and right circular polarisation transversally to $P$.

![Diagram](image)

**FIG. 9**: Coordinate basis with z-axis collinear to $P$ in the c.m.s. system of incoming protons $h_{1,2}$.

In the c.m.s. frame, we choose the basis with collinear $n_3$ and $P$ 4-vectors, therefore:

$$P = (E, 0, 0, P_z), \quad n_1^\beta = (0, 1, 0, 0),$$

$$n_2^\beta = (0, 0, 1, 0), \quad n_3^\beta = \frac{1}{M}(|P|, 0, 0, E)$$ (3.17)

where $|P| = \sqrt{E^2 - M^2}$. $n_2$ is chosen to be transverse to the c.m.s. beam axis (see figure 9), while $n_1$ and $n_3$ are turned around by the polar angle $\psi = [0...\pi]$ between $P$ and the c.m.s. beam axis. In the polarisation basis $[n_1, n_2, n_3]$ we have the following coordinates of the incoming protons:

$$h_1 = \frac{\sqrt{s}}{2}(1, - \sin \psi, 0, \cos \psi), \quad h_2 = \frac{\sqrt{s}}{2}(1, \sin \psi, 0, - \cos \psi)$$ (3.18)

In the same basis, the gluon transverse momenta are written:

$$k_{1,t} = (0, K_{1,t}^x \cos \psi, K_{1,t}^y, K_{1,t}^z \sin \psi), \quad k_{2,t} = (0, K_{2,t}^x \cos \psi, -K_{1,t}^y, K_{2,t}^z \sin \psi)$$ (3.19)

where $K_{1,2,t}^x, \pm K_{1,2,t}^y$ are the components of the gluon transverse momenta in the basis with $z$-axis collinear to the c.m.s. beam axis (see figure 10). $K_{1,t}^y$ is necessarily opposite to $K_{2,t}^y$ because as $P$ is in the (x,y) plane with the momentum conservation: $0 = P^y = k_{1,t}^y + k_{2,t}^y = K_{1,t}^y + K_{2,t}^y$. 

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FIG. 10: Basis with z-axis collinear to the c.m.s. beam axis in the c.m.s. system. The green plane is the beam axis transverse plane.

From (3.17) and (3.18) we can derive the relation between the charmonium energy $E$, the polar angle $\psi$ and the covariant scalar products in the considered basis:

$$
E = \frac{p_1 \cdot P + p_2 \cdot P}{\sqrt{s}}, \quad \cos \psi = \frac{p_1 \cdot P - p_2 \cdot P}{\sqrt{s}|P|}, \quad \sin \psi = \frac{p_2 \cdot n_1 - p_1 \cdot n_1}{\sqrt{s}} \quad (3.20)
$$

Moreover using (2.2) and $P = k_{1,t} + k_{2,t}$ yields:

$$
x_1 = \frac{E + |P| \cos \psi}{\sqrt{s}}, \quad x_2 = \frac{E - |P| \cos \psi}{\sqrt{s}} \quad (3.21)
$$

Then using the momentum conservation $|P| \sin \psi = |P_t| = K_{1,t}^x + K_{2,t}^x$, and the decompositions of $k_{1,t} \cdot k_{2,t}$ and $k_{1,2,t}$ in their components $K_{1,2,t}^x, \pm K_{1,2,t}^y$ give:

$$
K_{1,t}^x = -\frac{k_{2,t}^2 + k_{1,t} \cdot k_{2,t}}{|P| \sin \psi}, \quad K_{2,t}^x = -\frac{k_{2,t}^2 + k_{1,t} \cdot k_{2,t}}{|P| \sin \psi},
K_{1,t}^y = \frac{\sqrt{k_{1,t}^2 k_{2,t}^2 - (k_{1,t} \cdot k_{2,t})^2}}{|P_t|} \text{sign}(K_{1,t}^y) \quad (3.22)
$$

Inserting all the explicit terms (3.16), (3.17), (3.18), (3.19), (3.20), (3.21) and (3.22) in the (3.15) $J = 1$ vertex factor, we obtain the vertex factor in the c.m.s. frame using Mathematica:

$$
V_{j=1,\lambda}^{\text{cO}_1,\text{cO}_2} = -2g_s^2 R'(0) R_{\text{cO}_1,\text{cO}_2} \sqrt{\frac{6}{\pi M N_c}} \left[ \frac{1}{\sqrt{2}} \left( i |\lambda| (k_{1,t}^2 - k_{2,t}^2)(k_{1,t} \cdot k_{2,t}) \text{sign}(\sin \psi) M^2 + \lambda (k_{1,t}^2 + k_{2,t}^2) |k_{1,t} \times k_{2,t}| \times n_t |M_t^2 \text{sign}(K_t^y) \text{sign}(\cos \psi) \right) + (1 - |\lambda|)(k_{1,t}^2 + k_{2,t}^2) |k_{1,t} \times k_{2,t}| \times n_3 |M_t^2 \text{sign}(K_t^y) \text{sign}(\sin \psi) \right]. \quad (3.23)
$$
where
\[
||k_{1,t} \times k_{2,t} \times n_1|| = \sqrt{k_{1,t}^2 k_{2,t}^2 - (k_{1,t} \cdot k_{2,t})^2} |\cos \psi|,
\]
\[
||k_{1,t} \times k_{2,t} \times n_3|| = \frac{E}{M} \sqrt{k_{1,t}^2 k_{2,t}^2 - (k_{1,t} \cdot k_{2,t})^2} |\sin \psi|
\]
(3.24)

Inspecting relation (3.23) we notice that if \(k_{1,t} \to 0\) or \(k_{2,t} \to 0\) then \(V_{J=1,\lambda}^{\gamma_{12}} \to 0\). Which means that gluon transverse momenta are necessary to get a non zero cross section.

3. The amplitude \(V_{J=2,\lambda}^{\gamma_{12}}\)

Inserting (3.8), (3.8), (3.11), (3.12) and (3.13) in (3.7) we obtain with Mathematica the vertex factor for \(J = 2\):

\[
V_{J=2,\lambda}^{\gamma_{12}} = 2ig_s^2 \frac{R'(0)\delta^{\gamma_{12}}}{MM T(k_1 \cdot k_2)} \sqrt{\frac{3}{\pi MNc}} \varepsilon^{(\lambda)}_{\rho\alpha} \left[(k_{1,t} \cdot k_{2,t})(k^\alpha_{1,t} - k^\alpha_{2,t}) + (x_1 h_1^\alpha - x_2 h_2^\alpha)M^2 - (k^\rho_{1,t} - k^\rho_{2,t})M^2\right] - 2(k_1 \cdot k_2) \left(M^2(k^\rho_{1,t} k^\rho_{2,t} + k^\rho_{1,t} h^\rho_{2,t}) - k^2_{1,t}(k^\alpha_{1,t} h^\alpha_{2,t} + k^\alpha_{1,t} k^\alpha_{2,t}) - h^\alpha_{1,t}(k^\alpha_{1,t} h^\alpha_{2,t} + k^\alpha_{1,t} k^\alpha_{2,t}) + (x_1 h_1^\alpha - x_2 h_2^\alpha)h_2^\alpha_1, k_1 \cdot k_2)\right)
\]
(3.25)

This result is equivalent to the one in [22] but without neglecting the terms in \(x_1^2 \simeq x_2^2 \simeq 0\) which are non negligible at large \(P_T\). As for \(J = 1\), in expression (3.25) the indices still appear; to obtain a free-index expression we need a specific frame in order to use explicit 4-vectors.

The polarisation tensor \(\varepsilon^{(\lambda)}_{\rho\alpha}\) can be written in terms of \([n_1, n_2, n_3]\) basis in the following representation [22]:

\[
\varepsilon^{(\lambda)}_{\rho\alpha} = \frac{\sqrt{6}}{12} (2 - |\lambda|)(1 - |\lambda|) [g_{\rho\alpha} - \frac{P_\rho P_\alpha}{M^2}] + \frac{\sqrt{6}}{4} (2 - |\lambda|)(1 - |\lambda|) n^\rho_3 n^\alpha_3 + \frac{1}{4} |\lambda|(1 - |\lambda|)[n^\rho_1 n^\alpha_1 - n^\rho_2 n^\alpha_2] + \frac{1}{4} |\lambda|(1 - |\lambda|)[n^\rho_1 n^\alpha_2 + n^\rho_2 n^\alpha_1] + \frac{1}{2} |\lambda|(2 - |\lambda|)[n_3^\rho n^\alpha_3 + n_3^\rho n^\alpha_2] + \frac{1}{2} |\lambda|(2 - |\lambda|)[n_2^\rho n^\alpha_2 + n_2^\rho n^\alpha_1] + \frac{1}{2} |\lambda|(2 - |\lambda|)[n_1^\rho n^\alpha_1 + n_1^\rho n^\alpha_2]
\]
(3.26)

As we did previously for \(J = 1\), inserting all the explicit terms (3.26), (3.17), (3.18), (3.19), (3.20), (3.21), (3.22) and (3.24) in the (3.25) \(J = 2\) vertex factor, we obtain with Mathematica the free-index vertex factor in the c.m.s. frame:
Finally the charmonium transverse momentum is:

\[ V_{\gamma_c \gamma^*}^{c=2, \lambda} = 2i g_s^2 \sqrt{\frac{1}{3\pi MNc}} M|P_t|^2 (M^2 - k_{1,t}^2 - k_{2,t}^2) \frac{R'(0) \delta c^2}{M|P_t|^2 (M^2 - k_{1,t}^2 - k_{2,t}^2)} \left[ 6M^2 i|\lambda|(k_{1,t}^2 - k_{2,t}^2) \operatorname{sign}(K_{\gamma}^2) \times \left( |[k_{1,t} \times k_{2,t}] \times n_1|(1 - |\lambda|) \operatorname{sign}(|\sin(\psi)|) \operatorname{sign}(\cos(\psi)) + |[k_{1,t} \times k_{2,t}] \times n_3|(2 - |\lambda|) \right) - 
- [2k_{1,t}^2 k_{2,t}^2 + (k_{1,t}^2 + k_{2,t}^2)(k_{1,t} \cdot k_{2,t})] \left( 3M^2(\cos^2 \psi + 1) \lambda(1 - |\lambda|) + 
+ 6ME \sin(2\psi) \lambda(2 - |\lambda|) \operatorname{sign}(\sin(\psi)) \operatorname{sign}(\cos(\psi)) + \sqrt{6}(M^2 + 2E^2) \sin^2 \psi(1 - |\lambda|)(2 - |\lambda|) \right) \right] \tag{3.27} \]

Once again inspecting relation (3.27) we notice that if \( k_{1,t} \rightarrow 0 \) or \( k_{2,t} \rightarrow 0 \) then \( V_{\gamma_c \gamma^*}^{c=2, \lambda} \rightarrow 0 \). Which means that gluon transverse momenta are necessary to get a non zero cross section.

### D. Rapidity dependence of \( g^* g^* \rightarrow \chi_c^J \) amplitudes squared

As a first step in polarisation analysis, it is interesting to study the \( y \)-dependence of the different hard process amplitude squared \( |V_J|^2 \) for the \( \chi_c \) total angular momenta \( J = 1, 2 \) and their corresponding helicities \( \lambda = 0, \pm 1 \) and \( \lambda = 0, \pm 1, \pm 2 \). It is convenient to express them in terms of transverse 3-momentum of the gluons \( (k_{1,t} = -k_{1,t} \) and \( k_{2,t} = -k_{2,t} \) and the angle between them \( \phi \) in the c.m.s. frame of colliding nucleons with the z-axis along the \( \chi_c \) momentum \( P \) (see figure 9).

We first look to the relations between physical variable sets \( (\psi, k_{1,t}, k_{2,t}) \) and \( (y, \phi, k_{1,t}, k_{2,t}) \).

By using the rapidity definition (2.12) and (3.21) the longitudinal fractions are:

\[ x_1 = \frac{M_T \cosh y}{\sqrt{s}}, \quad x_2 = \frac{M_T \sinh y}{\sqrt{s}} \tag{3.28} \]

Then the decomposition of \( P \) into longitudinal and transverse components, and (2.1):

\[ |P|^2 = |P_0|^2 + |P_t|^2 \]
\[ = |h|^2(x_1 - x_2)^2 + |P_t|^2 \]
\[ = M_T \sinh y^2 + |P_t|^2 \tag{3.29} \]

Where \( h = h_1 = -h_2 \) in the c.m.s. frame of colliding nucleons. Then with \( \sin \psi = \pm |P_t|/|P| \), \( \cos \psi = \pm |P_0|/|P| \), \( E^2 = |P|^2 + M^2 \) (with \( E \) the charmonium energy) and (3.29) we obtain:

\[ \sin \psi = \pm \sqrt{\frac{1 - \frac{M^2}{|P|^2} \sinh y^2}{1 + \sinh y^2}}, \quad \cos \psi = \frac{M_T \sinh y^2}{|P|}, \quad E = M_T \cosh y \tag{3.30} \]

Finally the charmonium transverse momentum is:

\[ |P_t|^2 = |k_{1,t}|^2 + |k_{2,t}|^2 + 2|k_{1,t}||k_{2,t}| \cos \phi \tag{3.31} \]

To normalize the amplitudes squared, we first compute the sums of the latter over helicities \( \lambda \), up to some constant factor \( N^J \). By using the relations in III D. and vertex factor expressions (3.23) and (3.27) we get:
where for compactness:

$$A = M_T^6(|k_{1,t}|^2 + |k_{2,t}|^2)^2 + (M^6(|k_{1,t}|^2 - |k_{2,t}|^2)^2 - M_T^6(|k_{1,t}|^2 + |k_{2,t}|^2)^2)$$
$$B = 4M^2M_T^2(|k_{1,t}|^4 + |k_{1,t}|^2|k_{2,t}|^2 + |k_{2,t}|^4) + (M^4 + M_T^4)(|k_{1,t}|^4 + 10|k_{1,t}|^2|k_{2,t}|^2 + |k_{2,t}|^4)$$
$$C = 8(M^4 + M_T^4)(|k_{1,t}|^2 + |k_{2,t}|^2)^2 - 2M^2M_T^2(|k_{1,t}|^4 - 4|k_{1,t}|^2|k_{2,t}|^2 + |k_{2,t}|^4)$$

As you can see in (3.32), the sums $S^{J=1,2}$ are $y$-independent. For $\chi_{c}^{J=1}$ the polarised squared amplitudes are:

$$|V_{\lambda=0}^{J=1}|^2 = S^{J=1} \frac{M_T^6(M^2 - M_T^2)(|k_{1,t}|^2 + |k_{2,t}|^2)^2 \tan^2 \phi \cosh^2 y}{A(M^2 - M_T^2 \cosh^2 y)}$$
$$|V_{\lambda=\pm 1}^{J=1}|^2 = S^{J=1} \frac{M_T^2}{2A} \left[ QM^4 - \frac{M_T^6(|k_{1,t}|^2 + |k_{2,t}|^2)^2 \tan^2 \phi \sinh^2 y}{M^2 - M_T^2 \cosh^2 y} \right]$$

For $\chi_{c}^{J=2}$ the polarised squared amplitudes are:

$$|V_{\lambda=0}^{J=2}|^2 = S^{J=2} \frac{(M^2 - M_T^2)^2 R(M^2 + 2M_T^2 \cosh^2 y)}{2(B + C \cos \phi + D \cos(2\phi))(M^2 - M_T^2 \cosh^2 y)^2}$$
$$|V_{\lambda=\pm 1}^{J=2}|^2 = S^{J=2} U \cosh^2 y \left[ Q(M^2 - M_T^2 \cosh^2 y) \sin^2 \phi - M_T^2 R^2 \sinh^2 y \right]$$
$$|V_{\lambda=\pm 2}^{J=2}|^2 = \frac{3M^4S^{J=2} R(M^2 - M_T^2 \cosh(2y))^2 - 4M_T^2 Q(M^2 - M_T^2 \cosh^2 y) \sin^2 \phi \sinh^2 y}{4(B + C \cos \phi + D \cos(2\phi))(M^2 - M_T^2 \cosh^2 y)^2}$$

where for compactness:

$$U = 6M^2M_T^4(M^2 - M_T^2)$$
$$R = 2(|k_{1,t}|^2|k_{2,t}|^2 + (|k_{1,t}|^2 + |k_{2,t}|^2)^2) \cos^2 \phi$$
$$Q = (|k_{1,t}|^2 - |k_{2,t}|^2)^2$$

Even though the expressions (3.33) and (3.34) look different than in [23], they give the same results. To study the rapidity dependence of the normalized squared amplitudes for different meson helicities, we arbitrarily fixed kinematical variables $|k_{1,t}| = |k_{2,t}| = 0.5 \text{ GeV}$ and $\phi = 0.2$.

As you can see in the figure 11, the total signal is dominated by maximal polarisation contributions, i.e. by $\lambda = \pm 1$ for $\chi_{c}^{J=1}$ and $\lambda = \pm 2$ for $\chi_{c}^{J=2}$. This result is expected for relativistic particles; indeed the long-dotted curve represents the kinematics of the meson: excepted the area near $y = 0$ the meson is relativistic (its mass is negligible compare to its energy) and can be considered as massless. Therefore except at small $y$ i.e. at large $P_t$, the non-maximal polarisation contributions might be considered as negligible.
FIG. 11: Meson rapidity dependence of the normalized hard process $g^* g^* \rightarrow \chi_c^{J=1}$ (left figure) and $g^* g^* \rightarrow \chi_c^{J=2}$ (right figure) amplitudes squared for different meson helicities $\lambda$ at arbitrarily fixed kinematical variables $|k_1| = |k_2| = 0.5$ GeV and $\phi = 0.2$. The long dashed line which does not correspond to the axis name, represents the kinematics of the particle: in the small value area the particle is non relativistic and high value areas the particle is relativistic and therefore considered as massless.

E. Unpolarised $J/\psi$ differential cross sections from $\chi_c$ production

It is important before going to the polarisation study, to first see if our formalism and its translation into FORTRAN codes give correct results in the unpolarised case.

By a change of variables, the general unpolarised $\chi^J_c$ production cross section in the $k_{\perp}$ factorisation approach (2.11) can be re-written as:

$$\sigma_{J\lambda}(pp \rightarrow \chi_c X) = \frac{1}{8(N_c^2 - 1)} \int d^2P_t dy |k_{1,t}| |k_{2,t}| \delta^2(k_{1,t} + k_{2,t} - P_t) |k_{1,t}| P_t \times$$

$$\times \mathcal{F}(x_1, k^2_{1,t}, \mu_1^2) \frac{V_{J,\lambda}^* V_{J,\lambda}}{(k^2_{1,t})^2 (k^2_{2,t})^2} \mathcal{F}(x_2, k^2_{2,t}, \mu_2^2)$$

(3.35)

where $y$ is the rapidity of the produced charmonium, $\phi_{k_{1,t}}$ is the angle between $k_{1,t}$ and the $x$-axis (integrated over $[0,2\pi]$). The $\mu_{1,2}^2$ are the scales of the unintegrated gluon distribution functions (UGDF) $\mathcal{F}(x_{1,2}, k^2_{1,2,t}, \mu_{1,2}^2)$ for the gluon 1 and 2 (also called $\mu^2$ in fig. 5, see IB 2. for more details). These scales are not precisely known but can be taken equal to the gluon virtualities (or virtual masses) $k^2_{1,2,t}$, the charmonium transverse mass squared $M^2_t$ or the half of the charmonium transverse mass (one for each gluon), squared $M^2_t/4$. The latter will turn out to give the best results with the thirteen UGDFs we have used in this work (found in [40]: the KMR UGDG [41] which is a mix between the DGLAP and BFKL evolutions, and the CCFM UGDFs [19] (old set JS2001, J2003 set 1, J2003 set 2, J2003 set 3, set A0, set A0+,set A0-, set A1, set B0, set B0+, set B0- and set B1). These UGDFs are among the ones that are suitable for the $k_{\perp}$ factorisation approach. Similarly to the unknown scales $\mu_{1,2}^2$, the choice of the UGDF is a theoretical uncertainty that will be depicted by a UGDFs error band in our predictions.
As seen in fig. 2, the $J/\psi$ state can be indirectly produced by a radiative decay of the $\chi^J_c$ states. The unpolarised $J/\psi$ production cross section from $\chi^J_c$ is:

$$\sigma^J(J/\psi \text{ from } \chi^J_c) = B_J(\chi^J_c \rightarrow J/\psi \gamma) \cdot \sum_{\lambda=-J}^{J} \sigma^J_{\lambda\lambda}(pp \rightarrow \chi_c X)$$

and more generally from $\chi_c$:

$$\sigma(J/\psi \text{ from } \chi_c) = \sum_{J=0}^{2} \sigma^J(J/\psi \text{ from } \chi^J_c)$$

where $B_J$ are the branching ratios of the $\chi^J_c \rightarrow J/\psi \gamma$ decays ($B_0 = 0.0117$, $B_1 = 0.344$ and $B_2 = 0.195$ [4]). The branching ratio for the muonic decay of $J/\psi$ is taken to be $B(J/\psi \rightarrow \mu^+\mu^-) = 0.0593$.

1. Unpolarised results for Tevatron Run I at energy $\sqrt{s} = 1.8$ TeV

We first look at the transverse momentum $P_t$ dependence of the differential cross section $d\sigma/dP_t(J/\psi \text{ from } \chi^J_c)$. The rapidity $y$ is integrated over the interval [-4,4] (greater extreme values give the same results) and $|k_{1,t}|$ is integrated over the interval [0.6,40] (not from zero to avoid a possible divergence of the term $1/((k_{1,t}^2)^2(k_{2,t}^2)^2)$; the choice here done for the minimum value has not a great influence on the final result). If the contrary is not specified, our results are computed with the UGDF scale choice $\mu^2 = M^2_t/4$.

To take into account the next-to-leading order (NLO) $gg \rightarrow \chi^J_c$ vertex diagrams, NLO correction factors $K_{NLO}$ [43], applied to the amplitudes, have been computed for $\chi^J_c = 0$ ($K_{NLO} = 1.30$) and $\chi^J_c = 2$ ($K_{NLO} = 0.79$), but not yet for $\chi^J_c = 1$ because of the Landau-Yang theorem. Depending on the value of the factor $K_{NLO} \chi^J_c = 1$, the choice of the best UGDF might be different. For example if $K_{NLO} > 1$, the UGDF CCFM B0 may be a better choice, as it slightly underestimates the data (see fig. 15), at the contrary if $K_{NLO} < 1$, the better choice may be CCFM J2003 set 1, as it slightly overestimates the data (see fig. 17).

In the data, from the CDF collaboration [44], we use below, the differential cross section depends on the transverse momentum of $J/\psi$. However our theoretical predictions have a dependence on the transverse momentum of $\chi_c$. The classical difference between the two, due to the radiation, is of the order of 300 Mev, i.e. $P_t(J/\psi) = P_t(\chi_c) + 300$MeV. Therefore, keeping this in mind, we will favour predictions that slightly underestimate the data. This preference is magnified by the fact that we do not know if the COM (see IB.1.) contribution is negligible or if it will increase the predicted values. Despite that we have to keep in mind that the COM prediction is determined by a choice of uncalculable parameters that give the best fit for the CSM+COM prediction, in Hägler et al. paper [2] one can observe that if the COM is not negligible it mainly increases the CSM prediction at large $P_t$. In the latter paper they use the Kwiecinski Martin Stasto (KMS) UGDF [42].

Although we do not have a perfect fit, we can reasonably say that we obtain better results with our UGDF choices (CCFM set B0 and J2003 set 1) and only with the CSM, than the two Hägler’s et al. predictions with the CSM and the CSM+COM (see fig.5 in [2]
FIG. 12: [LO, scale $\mu^2 = M_t^2/4$, pseudorapidity cut $|\eta| < 0.6$, $\sqrt{s} = 1.8$ TeV] The transverse momentum differential cross section predictions for the UGDF CCFM set B0 and the UGDFs error band in comparison to the prediction obtained by Hägler et al. [2] with the UGDF KMS and to the CDF data [44] for prompt $J/\psi$ (direct, from $\chi_c$ and $\Psi'$) and only for $\chi_c$ productions. 

**Analysis:** All the data points are within the UGDFs error band and the CCFM set B0 curve fits correctly the data with a slight underestimation at small $P_t$. The data show us that the $J/\psi$ from $\chi_c$ contribution to the prompt $J/\psi$ production is around 25-30% at this energy.

FIG. 13: [NLO, scale $\mu^2 = M_t^2/4$, pseudorapidity cut $|\eta| < 0.6$, $\sqrt{s} = 1.8$ TeV] Same as in fig.12 but with the next-to-leading order coefficients applied to the UGDF CCFM set B0 and the UGDFs error band predictions. 

**Analysis:** The NLO factor for $\chi_c^{J=2} (K_{NLO} = 0.79)$ being lesser than 1, it tends to slightly decrease the predicted values.
FIG. 14: [NLO, scale $\mu^2 = M_t^2/4$, pseudorapidity cut $|\eta| < 0.6$, $\sqrt{s} = 1.8$ TeV] Same as in fig.13, but with the explicit contributions of $\chi_{cJ=0}^c$, $\chi_{cJ=1}^c$ and $\chi_{cJ=2}^c$. Analysis: $\chi_{cJ=1}^c$ gives the higher contribution to the $J/\psi$ production from $\chi_c$. The yet unknown value of the $K_{NLO}$ factor for $\chi_{cJ=1}^c$ will therefore have a great influence on the prediction. The contribution of $\chi_{cJ=0}^c$ can be neglected (it comes of course from the small value of branching ratio $B_0$).

FIG. 15: [NLO, scale $\mu^2 = M_t^2/4$, pseudorapidity cut $|\eta| < 0.6$, $\sqrt{s} = 1.8$ TeV] Same as in fig.13 but with an additional 300 MeV to $P_t$ (see text) to go from $P_t(\chi_c)$ to $P_t(J/\psi)$. Analysis: The additional 300 MeV improves slightly the prediction from fig.14, but there is still an underestimation at small $P_t$. 

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FIG. 16: [NLO, scale $\mu^2 = M_t^2/4$, pseudorapidity cut $|\eta| < 0.6$, $\sqrt{s} = 1.8$ TeV] Same as in fig.13 but with the UGDF CCFM J2003 set 1. Analysis: If we assume that the COM contribution is negligible, a better UGDF choice is CCFM J2003 set 1 as it gives a better fit to the data and particularly at low $P_t$.

FIG. 17: [NLO, scale $\mu^2 = M_t^2/4$, pseudorapidity cut $|\eta| < 0.6$, $\sqrt{s} = 1.8$ TeV] Same as in fig.16 but with an additional 300 MeV to $P_t$ (see text) to go from $P_t(\chi_c)$ to $P_t(J/\psi)$. Analysis: With the additional 300 MeV to $P_t$ for CCFM J2003 set 1 gives clearly a better fit to the data in comparison to CCFM set B0 (fig. 15).
for the latter).

We then look at the rapidity $y$ dependence of the differential cross section $d\sigma/dy(J/\psi$ from $\chi^J_{\psi})$. The transverse momentum $P_t$ is integrated over the interval $[0, 20]$ and $|k_{1,t}|$ is integrated over the interval $[0.6, 40]$.

![Image](image_url)

**FIG. 18**: [NLO, scale $\mu^2 = M_t^2/4$, $P_t < 20$ GeV, $\sqrt{s} = 1.8$ TeV] The rapidity differential cross section predictions for the UGDFs CCFM set B0 and J2003 set 1, and the UGDFs error band. 

*Analysis*: No $y$ dependence has been experimentally studied at Tevatron energies. The curves are symmetric with the $y = 0$ axis as expected because of the symmetry system around $y = 0$.

2. **Unpolarised results for Tevatron Run II at energy $\sqrt{s} = 1.96$ TeV**

As in the previous section, we look at the transverse momentum $P_t$ dependence of the differential cross section $d\sigma/dP_t(J/\psi$ from $\chi^J_{\psi})$. The rapidity $y$ is integrated over the interval $[-4,4]$ and $|k_{1,t}|$ is integrated over the interval $[0.6,40]$.

3. **Unpolarised results for LHC Run I at energy $\sqrt{s} = 7$ TeV**

As in the previous sections, we look at the transverse momentum $P_t$ dependence of the differential cross section $d\sigma/dP_t(J/\psi$ from $\chi^J_{\psi})$ in fig. 20. The rapidity $y$ is integrated over the interval $[-0.75,0.75]$ and $|k_{1,t}|$ is integrated over the interval $[0.6,40]$.

We then look at the rapidity $y$ dependence of the differential cross section $d\sigma/dy(J/\psi$ from $\chi^J_{\psi})$ in fig. 21. The transverse momentum $P_t$ is integrated over the interval $[0,20]$ and $|k_{1,t}|$ is integrated over the interval $[0.6,40]$.
FIG. 19: [NLO, scale $\mu^2 = M^2/4$, pseudorapidity cut $|\eta| < 0.6$, $\sqrt{s} = 1.96$ TeV] The transverse momentum differential cross section predictions for the UGDF CCFM set B0 and the UGDFs error band in comparison to the CDF data [45] for prompt $J/\psi$ (direct, from $\chi_c$ and $\Psi'$). 

Analysis: The predicted $J/\psi$ from $\chi_c$ contribution to the prompt $J/\psi$ data points is around 15%, which is smaller than the 25-30% we had in fig. 12.

4. Unpolarised results for LHC Run II at energy $\sqrt{s} = 14$ TeV

As in the previous sections, we look at the transverse momentum $P_t$ dependence of the differential cross section $d\sigma/dP_t(J/\psi$ from $\chi_c^J)$ in fig. 22. The rapidity $y$ is integrated over the interval $[-0.75,0.75]$ and $|k_{1,t}|$ is integrated over the interval $[0,40]$.

We then look at the rapidity $y$ dependence of the differential cross section $d\sigma/dy(J/\psi$ from $\chi_c^J)$ in fig. 23. The transverse momentum $P_t$ is integrated over the interval $[0,20]$ and $|k_{1,t}|$ is integrated over the interval $[0,6]$.

We have therefore seen that the above results for unpolarised $J/\psi$ production are successfully compared to the previous calculations in the literature and to the experimental data on prompt $J/\psi$ production from radiative $\chi_c$ decays at Tevatron $\sqrt{s} = 1.8$ TeV. We confirm that $k_\perp$ factorisation framework leads to good description of the experimental data without the color octet contribution at the leading order of perturbation theory. Additionally we have made predictions for the differential cross sections (in meson transverse momentum and rapidity) for the unpolarised $\chi_c$ and $J/\psi$ (from $\chi_c$ decays) production at the LHC energies (7 TeV and 14 TeV). The following sections are dedicated to the polarisation studies: the first step being the polarisation of $\chi_c$ (study of the angular distribution of $J/\psi$, its decay product) and the second step the polarisation of $J/\psi$ from $\chi_c$ radiative decay (study of the muonic angular distribution).
FIG. 20: [NLO, scale $\mu^2 = M^2_t/4$, $|y| < 0.75$, $\sqrt{s} = 7$ TeV] The transverse momentum differential cross section predictions for the UGDF CCFM set B0 ($J/\psi$ from $\chi^J_c$) and the UGDFs error band in comparison to the prediction obtained by Saleev [47] with the UGDF KMR and to the Atlas data [46] for $J/\psi$ prompt (direct, from $\chi_c$ and $\psi'$). **Analysis:** Our CCFM set B0 curve fits correctly the prediction from Saleev paper which gave a good fit to the data when the contributions from direct $J/\psi$ and $\psi'$ are added. The predicted $J/\psi$ from $\chi_c$ contribution to the prompt $J/\psi$ data points is around 10-20%.

IV. ANGULAR DISTRIBUTION OF $J/\psi$ MESON FROM $\chi^J_c$ DECAY

Now that we have computed the hard amplitude $g^* g^* \rightarrow \chi^J_c$, the next step is to deal with the decay process $\chi^J_c \rightarrow J/\psi + \gamma$. The decay into two spinning particles is developed in [33] and summarized in the following.

A. Decay into two spinning particles

As illustrated in figure 24, let $\theta$ and $\phi$ be the polar and azimuthal angles of the $J/\psi$ meson in the $\chi^J_c$ rest frame (called helicity frame).

Based on figure 25, the differential cross-section of the $J/\psi$ production from $\chi^J_c$ decay can be written as a product of the $\chi^J_c$ production polarised cross section, the branching ratio and angular distribution of $\chi^J_c \rightarrow J/\psi \gamma$:

$$\frac{d\sigma^{J}_{J/\psi}}{d\Omega} = B_J(\chi^J_c \rightarrow J/\psi \gamma) \cdot \sigma^{J}_{\chi_c} \cdot W^{J}(\theta, \phi), \text{ where } W^{J}(\theta, \phi) = \sum_{\lambda,\lambda'}^{J} \rho^{J}_{\lambda\lambda'} A^{J}_{\lambda\lambda'}(\theta, \phi) \ (4.1)$$

where $\sigma^{J}_{\chi_c}$ is the sum of the $\chi^J_c$ production cross sections $\sigma^{J}_{\lambda\lambda}$ (whose expression is (2.11))
FIG. 21: [NLO, scale $\mu^2 = M_t^2/4$, $P_t < 20$ GeV, $\sqrt{s} = 7$ TeV] The rapidity differential cross section predictions for the UGDFs CCFM set B0 and J2003 set 1, and the UGDFs error band in comparison to ALICE data [48] for prompt $J/\psi$. Analysis: The $y$ unpolarised prediction seems to be correct. Indeed the predicted $J/\psi$ from $\chi_c$ contribution to the prompt $J/\psi$ data points is, as it was in fig. 20, around 10-20% (the points corresponding to 13% of the prompt values have been added to show you this) and moreover the evolution of the curves fits the data one.

FIG. 22: [NLO, scale $\mu^2 = M_t^2/4$, $|y| < 0.75$, $\sqrt{s} = 14$ TeV] The transverse momentum differential cross section predictions for the UGDF CCFM set B0 ($J/\psi$ from $\chi_c$), its UGDFs error band and the $J = 1$ and $J = 2$ contributions. Analysis: These predictions are to be compared with future LHC experiments at energy $\sqrt{s} = 14$ TeV.

over its $\chi_c^J$ polarisation, i.e.:

$$\sigma_{\chi_c}^J = \sum_{\lambda=-J}^{J} \sigma_{\lambda\lambda}^J$$

and $d\Omega = \sin \theta d\theta d\phi$ is the element of the solid angle and $B_J$ the branching ratio of the
FIG. 23: [NLO, scale $\mu^2 = M_T^2/4$, $\sqrt{s} = 14$ TeV] The rapidity differential cross section predictions for the UGDFs CCFM set B0 and J2003 set 1 and the UGDFs error band. Analysis: These predictions are to be compared with future LHC experiments at energy $\sqrt{s} = 14$ TeV.

FIG. 24: $J/\psi$ momentum in the $\chi_c^j$ rest frame. The production plane, i.e. the plane with colliding hadrons and $\chi_c^j$ momentum, is $(x, z)$ (as in figure 10). The choice of the direction of the $z$ axis will be discussed in IV C.

$\chi_c^j \rightarrow J/\psi \gamma$ decay. $\rho_{\lambda\lambda'}$ is the integrated hermitian density matrix corresponding to $\chi_c^j$ production process. It is defined as follows and has the following properties:

$$\rho^J_{\lambda\lambda'} \equiv \frac{\sigma^J_{\lambda\lambda'}}{\sigma^J_{\chi_c}}, \quad \rho^J_{\lambda\lambda'} = \rho^J_{\lambda'\lambda}, \quad \rho^J_{-\lambda,-\lambda'} = (-1)^{\lambda' - \lambda} \rho^J_{\lambda\lambda'}, \quad \sum_{\lambda=-J}^{J} \rho^J_{\lambda\lambda} = 1 \quad (4.3)$$

The first property is the hermitian property of the density matrix coming from $(V_{J,\lambda} V_{J,\lambda}^*) = V_{J,\lambda'} V_{J,\lambda'}^*$ in (2.11). The second property corresponds to the invariance of the amplitude under reflection in the production plane by the action of the operator $\exp(-i\pi J_y)P$, where $P$ is the parity operator. The last property is a direct consequence of
FIG. 25: Basic diagram for decay cross section calculation. $\lambda_1 = 0, \pm 1$ is the $J/\psi$ helicity and $\lambda_2 = \pm 1$ is the $\gamma$ helicity. The photon $\gamma$ moves in the opposite direction to $J/\psi$ i.e. $(\pi - \theta, \pi + \phi)$.

its definition and of (4.2).

The function $W^J(\theta, \phi)$ in equation 4.1 is the angular distribution of the $J/\psi$ meson, and its integration over the solid angle is equal to unity, and $A^J\lambda_\lambda_{\lambda_1, \lambda_2} (\theta, \phi)$ is the decay angular distribution given by the normalized product of the decay transition matrix $T$:

$$A^J\lambda_\lambda_{\lambda_1, \lambda_2} (\theta, \phi) = \sum_{\lambda_1, \lambda_2} T^J|\lambda_1, \lambda_2 \rangle (\theta, \phi) \cdot T^J|\lambda_1, \lambda_2 \rangle (\theta, \phi)^* \sum_{\lambda_1, \lambda_2} |T^J|\lambda_1, \lambda_2 \rangle (\theta, \phi)|^2 d\Omega \quad (4.4)$$

where the transition matrix element $T^J|\lambda_1, \lambda_2 \rangle (\theta, \phi)$ of the radiative decay process $\chi_c^J(\lambda) \rightarrow J/\psi(\lambda_1, \theta, \phi) + \gamma(\lambda_2, \pi - \theta, \phi + \pi)$, in the considered frame of reference, can be decomposed into a decay amplitude $T^J|\lambda_1, \lambda_2 \rangle$ and a representation of the rotation group $D$ that describes the change in quantization orientations from the state $|J, \lambda \rangle$ to $|\lambda_1, \lambda_2, \theta, \phi \rangle$:

$$T^J|\lambda_1, \lambda_2 \rangle (\theta, \phi) = \langle \lambda_1, \lambda_2, \theta, \phi | T | J, \lambda \rangle = \sqrt{\frac{2J + 1}{4\pi}} T^J|\lambda_1, \lambda_2 \rangle D^{J*}_{\lambda_1, \lambda_2} (-\phi, \theta, \phi),$$

$$T^J|\lambda_1, \lambda_2 \rangle = \sqrt{\frac{4\pi}{2J + 1}} \langle \lambda_1, \lambda_2, 0, 0 | T | J, \lambda \rangle |_{\lambda_1 - \lambda_2} \quad (4.5)$$

The parity (symmetry of the field through space inversion) of a particle is given by the products of its intrinsic parities (1 for a charged fermion and $-1$ for an charged anti-fermion) and of its extrinsic parity (equal to $(-1)^L$). Let $\eta = 1$, $\eta_1 = -1$ and $\eta_2 = -1$ be respectively the parities of $\chi_c(J)$, $J/\psi(J_1 = 1)$ and $\gamma(J_2 = 1)$. Due to parity conservation the decay amplitude $T^J|\lambda_1, \lambda_2 \rangle$ satisfies the following polarisation symmetry:

$$T^J_{-\lambda_1, -\lambda_2} = \eta_1 \eta_2 (-1)^{J_1 + J_2 - J} T^J_{\lambda_1, \lambda_2} = (-1)^J T^J_{\lambda_1, \lambda_2} \quad (4.6)$$

The possible $J/\psi$ and $\gamma$ polarisation states are not only limited by the common rules $|\lambda_1| \leq J_1$ and $|\lambda_2| \leq J_2$ but also by the angular momentum conservation that imposes $|\lambda_1 - \lambda_2| \leq J$. Indeed as $J/\psi$ and $\gamma$ go in opposite directions, the polarisation of the system $[J/\psi; \gamma]$ is $|\lambda_1 - \lambda_2|$. Therefore, it follows by using (4.6) that we can define:
\[ J = 0 : \quad t_0^0 \equiv T_{1,1}^0 = T_{-1,-1}^0 \]  
\[ J = 1 : \quad t_1^1 \equiv T_{1,1}^1 = -T_{-1,-1}^1, \quad t_1^1 \equiv T_{0,-1}^1 = -T_{0,1}^1 \]  
\[ J = 2 : \quad t_2^0 \equiv T_{1,1}^2 = T_{-1,-1}^2, \quad t_2^1 \equiv T_{0,-1}^2 = T_{0,1}^2 \] (4.7)

Writing \( \alpha = -\phi, \beta = \theta \) and \( \gamma = \phi \) the Euler angles between the two quantization orientations, and also \( m' = \lambda \) or \( \lambda' \) and \( m = \lambda_1 - \lambda_2 \) the polarisations, the matrix element of the representation of the rotation group is:

\[
D_J^{m'm}(\alpha, \beta, \gamma) = \langle J, m'| D(\alpha, \beta, \gamma)| J, m \rangle \\
= \langle J, m' | \exp(-i\gamma J_z) \exp(-i\beta J_y) \exp(-i\alpha J_x)| J, m \rangle \\
= \exp(-i\gamma m') d_J^{m'm}(\beta) \exp(-i\alpha m) \tag{4.8}
\]

The \( d \)-functions \( d_J^{m'm}(\beta) \) may be evaluated from Wigner’s formula [34]:

\[
d_J^{m'm}(\beta) = \sum_{k = \max(0, m-m')}^{\min(J+m, J-m')} (-1)^{k-m+m'} \frac{\sqrt{(J+m)!(J-m)!(J+m')!(J-m')!}}{k!(k-m+m')!(J+m-k)!(J-m'-k)!} \times \\
\times (\cos \frac{\beta}{2})^{2J-m-m'-2k} (\sin \frac{\beta}{2})^{2k-m+m'} \tag{4.9}
\]

The orthogonality condition:

\[
\int d\Omega D_J^{m'm'}(-\phi, \theta, \phi) D_J^{m''m''}(-\phi, \theta, \phi) = \frac{4\pi}{2j+1} \delta_{m'm''} \tag{4.10}
\]

provides the normalization of \( A_{\lambda\lambda'}^J \) and \( W^J(\theta, \phi) \) with (4.3):

\[
\int d\Omega \ A_{\lambda\lambda'}^J(\theta, \phi) = \delta_{\lambda\lambda'} \tag{4.11}
\]

\[
\int d\Omega \ W^J(\theta, \phi) = \sum_{\lambda, \lambda' = -J}^J \rho_{\lambda\lambda'}^J \int d\Omega \ A_{\lambda\lambda'}^J(\theta, \phi) = \sum_{\lambda, \lambda' = -J}^J \rho_{\lambda\lambda'}^J \delta_{\lambda\lambda'} = 1
\]

With (4.11), if one integrates (4.1) over the solid angle, one obtains the narrow-width approximation that will be used in the unpolarised studies:

\[
\sigma_J^{J/\psi} = B_J(\chi_c J \rightarrow J/\psi + \gamma) \cdot \sigma_J^{\chi_c} \tag{4.12}
\]

B. Angular distribution of the \( J/\psi \) unpolarised production from \( \chi_c^J \)

Combining all the relations in IV A. with Mathematica, we get the angular distribution of the \( J/\psi \) production from \( \chi_c^J \):

\[
W^J=0(\theta, \phi) = \frac{\rho_{0,0}^0}{4\pi} \tag{4.13}
\]
The decay of $\chi_c^{J=0}$ does not have any angular dependence: it will not be relevant in future polarisation developments (where we will indeed study this angular dependence).

The angular distribution of the $J/\psi$ production from $\chi_c^{J=1}$:

$$W^{J=1}(\theta, \phi) = \frac{3}{4\pi} \left( \rho_{0,0}^1 [r_0^1 \cos^2 \theta + \frac{r_1^1}{2} \sin^2 \theta] + \rho_{1,1}^1 [r_0^1 \sin^2 \theta + \frac{r_1^1}{2} (1 + \cos^2 \theta)] - \sqrt{2} \sin(2\theta) (r_0^1 - \frac{r_1^1}{2}) [\text{Re}(\rho_{1,0}^1) \cos \phi - \text{Im}(\rho_{1,0}^1) \sin \phi] - \sin^2 \theta (r_0^1 - \frac{r_1^1}{2}) [\text{Re}(\rho_{1,-1}^1) \cos(2\phi) - \text{Im}(\rho_{1,-1}^1) \sin(2\phi)] \right)$$  (4.14)

and the angular distribution of the $J/\psi$ production from $\chi_c^{J=2}$:

$$W^{J=2}(\theta, \phi) = \frac{5}{4\pi} \left\{ \rho_{0,0}^2 \left[ \frac{1}{4} r_0^2 (3 \cos^2 \theta - 1)^2 + \frac{3}{2} r_1^2 \sin^2 \theta \cos^2 \theta + \frac{3}{8} r_2^2 \sin^4 \theta \right] + \rho_{1,1}^2 \left[ 3r_0^2 \sin^2 \theta \cos^2 \theta + \frac{r_1^2}{2} (4 \cos^4 \theta - 3 \cos^2 \theta + 1) + \frac{r_2^2}{2} \sin^2 \theta (\cos^2 \theta + 1) \right] + \frac{\sqrt{6}}{4} \sin(2\theta) (2r_0^2 (1 - 3 \cos^2 \theta) + 2r_1^2 \cos(2\theta) + r_2^2 \sin^2 \theta) [\text{Re}(\rho_{1,0}^2) \cos \phi - \text{Im}(\rho_{1,0}^2) \sin \phi] - \frac{1}{2} \sin^2 \theta \left( 6r_0^2 \cos^2 \theta + r_1^2 (\sin^2 \theta - 3 \cos^2 \theta) - r_2^2 \sin^2 \theta \right) [\text{Re}(\rho_{1,-1}^2) \cos(2\phi) - \text{Im}(\rho_{1,-1}^2) \sin(2\phi)] + \frac{1}{2} \sin^3 \theta (6r_0^2 - 4r_1^2 + r_2^2) \left[ \cos \theta [\text{Re}(\rho_{2,-1}^2) \cos(3\phi) - \text{Im}(\rho_{2,-1}^2) \sin(3\phi)] + \frac{1}{4} \sin \theta [\text{Re}(\rho_{2,-2}^2) \cos(4\phi) - \text{Im}(\rho_{2,-2}^2) \sin(4\phi)] \right] + \frac{\sqrt{6}}{4} \sin^2 \theta (2r_0^2 (3 \cos^2 \theta - 1) - 4r_1^2 \cos^2 \theta + r_2^2 (\cos^2 \theta + 1)) [\text{Re}(\rho_{2,0}^2) \cos(2\phi) - \text{Im}(\rho_{2,0}^2) \sin(2\phi)] - \frac{1}{4} \sin(2\theta) (6r_0^2 \sin^2 \theta + 4r_1^2 \cos^2 \theta - r_2^2 (\cos^2 \theta + 3)) [\text{Re}(\rho_{2,1}^2) \cos \phi - \text{Im}(\rho_{2,1}^2) \sin \phi] \right\}$$  (4.15)

where

$$r_X^J = \frac{|t_X^J|^2}{\sum_{X'}^J |t_X^J|^2}$$  (4.16)

are positive numbers to be determined experimentally, satisfying $\sum_{X=0}^J r_X^J = 1$. In Ref. [35] these values were extracted from the Fermilab E835 data on fractional amplitudes of the electric dipole (E1), magnetic quadrupole (M2), and electric octupole (E3) transitions (e.m. transitions that give a required parity change $\chi^+_c \rightarrow J/\psi^-$) in the exclusive reactions $p\bar{p} \rightarrow \chi_c J \rightarrow J/\psi \gamma \rightarrow e^+ e^-$, with $J = 1, 2$. They are listed here

$$r_0^1 = 0.498 \pm 0.032, \quad r_1^1 = 0.502 \pm 0.032,$$
$$r_0^2 = 0.075 \pm 0.029, \quad r_1^2 = 0.250 \pm 0.048, \quad r_2^2 = 0.674 \pm 0.052.$$  (4.17) (4.18)
The corresponding values for a pure E1 transition read $r_0^1 = r_1^1 = 0.5$, $r_0^2 = 0.1$, $r_1^2 = 0.3$, and $r_2^2 = 0.6$. Having these values we only need to calculate the independent components of $\rho^J_{\lambda\nu}$ to determine the $J/ψ$ angular distribution $W^J(θ, φ)$ completely.

C. About frames and polarisation

The polarisation of a particle is defined as the projection of its total angular momentum on a chosen axis (which is then said to be the polarisation axis). If this projection has in mean a preferential orientation, i.e. if the projection has a non-zero mean value, the particle is said to be polarised.

Quantum mechanics tells us that the choice of the polarisation axis is important: particle total angular momentum can be seen to be polarised with respect to a given axis while at the same time be unpolarised with respect to another one (the different projections do not commute: if one knows the polarisation along an axis, one can not know the polarisation along the other axis without changing the polarisation along the former axis). Usually the chosen axis is taken to be either collinear to the direction of one of the beams, or collinear to the momentum of the studied particle, or an axis geometrically derived from these two directions.

1. Polarisation frames

The polarisation (or coordinate) frames commonly used are [37]:

- Helicity frame (HX) (also called recoil frame): a frame where the charmonium is at rest, with the Z axis corresponding to the direction of the charmonium momentum in the laboratory frame, i.e. Z is the direction in which we have performed a Lorentz boost to switch from the laboratory frame to the charmonium rest frame.

- Gottfried-Jackson frame (GJ): a frame where the charmonium is at rest, with the Z axis corresponding to the momentum of one of the colliding hadrons in that frame.

- Target frame (T): a frame where the charmonium is at rest, with the Z axis corresponding to the momentum of one of the colliding hadrons (relatively to the one chosen in GJ) in that frame. In our case, where the two beams are identical, the T and GJ frames will give identical results (as they can be interchange).

- Collins-Soper frame (CS): a frame where the charmonium is at rest, with the Z axis corresponding to the bisector of the angle formed by the momentum of one colliding hadron and the opposite direction to the momentum of the other hadron.

Mathematically [36], to define the different coordinate systems we first need to define two auxiliary four-vectors $\tilde{A}$ and $\tilde{B}$ (the tilde notation means in this section that the vectors are defined in the charmonium rest frame, and without tilde it means that they are defined in
the center-of-mass (of colliding hadrons) frame) as:

$$\tilde{A}^\mu = A^\mu - \frac{A \cdot PP^\mu}{M^2}, \quad \text{where} \quad A = h_1 + h_2$$  \hspace{1cm} (4.19)

$$\tilde{B}^\mu = B^\mu - \frac{B \cdot PP^\mu}{M^2}, \quad \text{where} \quad B = h_1 - h_2$$  \hspace{1cm} (4.20)

where $P$ is the charmonium momentum and $h_1$ and $h_1$ are the colliding hadron momenta. These auxiliary four-vectors correspond to the "new" four-vector $\tilde{A}$ and $\tilde{B}$ after a Lorentz boost in the direction of the charmonium momentum $P$ from the center of mass frame to the charmonium rest frame.

![Production plane](image)

**FIG. 26:** Before the Lorentz boost: production plane in the center of mass frame

The general coordinate system is then defined as follow:

1. Choose $Z^\mu = \alpha_z \tilde{A}^\mu + \beta_z \tilde{B}^\mu$, with the normalisation $Z^2 = -1$.

2. Define $X^\mu = \alpha_x \tilde{A}^\mu + \beta_x \tilde{B}^\mu$ in the plane spanned by $\tilde{A}$, $\tilde{B}$, orthogonal to $Z$ and normalised by: $X \cdot Z = 0$, $X^2 = -1$.

3. Take $Y$ to complete a right-handed coordinate system in the charmonium rest frame,

$$Y^\mu = \frac{1}{M} \epsilon^{\mu\alpha\beta\gamma} P_\alpha X_\beta Z_\gamma$$

The four commonly used polarisation frames are then specified by the choice of $Z$. The description of the four frames we made above gives then:

- **HX frame:** $\tilde{Z} = -\tilde{A}/|\tilde{A}|$.
- **GJ frame:** $\tilde{Z} = \tilde{h}_1/|\tilde{h}_1|$
- **T frame:** $\tilde{Z} = -\tilde{h}_2/|\tilde{h}_2|$
CS frame: The Z axis bisects the angle between \( \hat{h}_1 \) and \( (-\hat{h}_2) \), i.e. 
\[
\hat{Z} \propto \hat{h}_1/|\hat{h}_1| + (-\hat{h}_2)/|\hat{h}_2|
\]

Then the covariant expressions for the coordinate axes are found from the following expressions for \( \alpha_{z,x}, \beta_{z,x} \): 

- HX frame: 
  \[
  \alpha_z = -\frac{M}{\sqrt{(A \cdot P)^2 - M^2 s}} \quad \beta_z = 0 \quad (4.21)
  \]
  \[
  \alpha_x = \frac{A \cdot P B \cdot P}{\sqrt{s ((A \cdot P)^2 - M^2 s)((A \cdot P)^2 - (B \cdot P)^2 - M^2 s)}}
  \]
  \[
  \beta_x = -\frac{\sqrt{(A \cdot P)^2 - M^2 s}}{\sqrt{s ((A \cdot P)^2 - (B \cdot P)^2 - M^2 s)}}
  \]

- GJ frame: 
  \[
  \alpha_z = \beta_z = \frac{M}{A \cdot P + B \cdot P} \quad (4.22)
  \]
  \[
  \alpha_x = -\frac{(B \cdot P)^2 + A \cdot P B \cdot P + M^2 s}{(A \cdot P + B \cdot P)\sqrt{s ((A \cdot P)^2 - (B \cdot P)^2 - M^2 s)}}
  \]
  \[
  \beta_x = \frac{(A \cdot P)^2 + A \cdot P B \cdot P - M^2 s}{(A \cdot P + B \cdot P)\sqrt{s ((A \cdot P)^2 - (B \cdot P)^2 - M^2 s)}}
  \]
\[ \alpha_z = -\beta_z = -\frac{M}{A \cdot P - B \cdot P} \tag{4.23} \]

\[ \begin{align*}
\alpha_x &= -\frac{(B \cdot P)^2 - A \cdot P B \cdot P + M^2 s}{(A \cdot P - B \cdot P) \sqrt{s ((A \cdot P)^2 - (B \cdot P)^2 - M^2 s)}} \\
\beta_x &= -\frac{(A \cdot P)^2 - A \cdot P B \cdot P - M^2 s}{(A \cdot P - B \cdot P) \sqrt{s ((A \cdot P)^2 - (B \cdot P)^2 - M^2 s)}}
\end{align*} \]

\[ \begin{align*}
\alpha_z &= -\frac{B \cdot P}{\sqrt{s ((A \cdot P)^2 - (B \cdot P)^2)}} \\
\beta_z &= \frac{A \cdot P}{\sqrt{s ((A \cdot P)^2 - (B \cdot P)^2)}} \\
\beta_x &= -\frac{M A \cdot P}{\sqrt{((A \cdot P)^2 - (B \cdot P)^2) ((A \cdot P)^2 - (B \cdot P)^2 - M^2 s)}} \\
\beta_z &= \frac{M B \cdot P}{\sqrt{((A \cdot P)^2 - (B \cdot P)^2) ((A \cdot P)^2 - (B \cdot P)^2 - M^2 s)}}
\end{align*} \tag{4.24} \]

Finally, the polarisation vectors are given by (a particular case for the HX frame was (3.16)):

\[\epsilon^\mu (\lambda = 0) = Z^\mu, \quad \epsilon^\mu (\lambda = \pm 1) = \frac{1}{\sqrt{2}} (\mp X^\mu - i Y^\mu). \tag{4.25} \]

2. Polarisation observables

Decaying particle polarisation plays an important role in the angular distribution of its decay products.

* If its polarisation is null then there is a spherical symmetry and the angular distribution of its decay products is isotropic.

* If the decaying particle is polarised and its total angular momentum is aligned to the chosen polarisation Z axis, then there is a cylindrical symmetry around the latter axis and the angular distribution of its decay products does not depend on the azimuthal angle \( \phi \) or \( \Phi \) but is dependent on the polar angle \( \theta \) or \( \Theta \) (see figure 24).

* More generally if the decaying particle is polarised but the total angular momentum is not collinear with the chosen polarisation Z axis, then the angular distribution is anisotropic and is dependent on the two polar and azimuthal angles. This fact underlines the importance of performing a measurement in two dimensions, since we do not know the charmonium polarisation axis a priori.
The angular distribution of the decay products can usually be written as:

\[ W(\theta, \phi) \propto 1 + \lambda_\theta \cos^2 \theta + \lambda_\phi \sin^2 \theta \cos(2\phi) + \lambda_{\theta\phi} \sin(2\theta) \cos \phi \quad (4.26) \]

where the three polarisation observables \( \lambda_\theta, \lambda_\phi, \) and \( \lambda_{\theta\phi} \) are measured experimentally and depend in the theoretical analysis on the density matrix elements \( \rho_{J,\lambda}^J \) as we will see for example in the \( \chi^J_c \) polarisation analysis (IV D). Moreover a charmonium state with total angular momentum \( J \) can be decomposed in a linear combination of eigenstates of the \( J_z \) operator [38]:

\[ |\text{Charmonium}\rangle = \sum_{\lambda=-J}^J \rho_{J,\lambda}^J |J,\lambda\rangle \]

where the \( \rho_{J,\lambda}^J \) are related to the production mechanism where the produced charmonium has a total angular momentum \( J \) and a \( Z \) axis polarisation \( \lambda \).

If the charmonium is produced with a fully \( Z \) axis polarisation \( \lambda = 0 \), one expects that only the corresponding \( \rho_{1,0}^J \) is non zero. One can therefore easily deduce the corresponding values of the observables \( \lambda_\theta, \lambda_\phi, \) and \( \lambda_{\theta\phi} \) that we be measured experimentally.

In the case of a \( J = 1 \) charmonium we have classically (see (4.31) or (5.20)):

\[ \lambda_\theta = -\frac{\rho_{1,1}^1 - \rho_{1,0}^1}{3\rho_{1,1}^1 + \rho_{1,0}^1} \quad (4.27) \]

If the charmonium is produced in a fully "longitudinal" polarisation state \( \lambda = 0 \) then the only non zero density matrix element is \( \rho_{1,0}^0 \) and with (4.27) one will expect to measure \( \lambda_\theta = 1.0 \). If the charmonium is instead produced in a fully "transverse" polarisation state \( \lambda = 1 \) or \( \lambda = -1 \) then the only non zero density matrix element is \( \rho_{1,1}^1 \) (\( \rho_{1,1}^{1,-1} \) with the density matrix property (4.3)) and with (4.27) one will expect to measure \( \lambda_\theta = -1/3.0 \). If the charmonium is unpolarised it means that \( \rho_{1,0}^0 \) and \( \rho_{1,1}^1 \) have to be equal and therefore one will expect to measure \( \lambda_\theta = 0.0 \), i.e. the decay products distribution does not depend on \( \cos^2 \theta \).

In the case of a \( J = 2 \) charmonium we have classically (see (4.34)):

\[ \lambda_\theta = -\frac{3\rho_{0,0}^2 - 3\rho_{1,1}^2 + 6\rho_{2,2}^2}{5\rho_{0,0}^2 + 9\rho_{1,1}^2 + 6\rho_{2,2}^2} \quad (4.28) \]

If the charmonium is produced in a fully "longitudinal" polarisation state \( \lambda = 0 \) then the only non zero density matrix element is \( \rho_{0,0}^0 \) and with (4.28) one will expect to measure \( \lambda_\theta = -3/5.0 \). If the charmonium is instead produced in a fully "transverse" polarisation state \( \lambda = 1 \) or \( \lambda = -1 \) then the only non zero density matrix element is \( \rho_{1,1}^1 \) (\( \rho_{1,1}^{1,-1} \) and with (4.28) one will expect to measure \( \lambda_\theta = -1/3.0 \). If the charmonium is instead produced in a fully "transverse" polarisation state \( \lambda = 2 \) or \( \lambda = -2 \) then the only non zero density matrix element is \( \rho_{2,2}^1 \) (\( \rho_{2,2}^{1,-1} \)) and with (4.28) one will expect to measure \( \lambda_\theta = 1.0 \). If the charmonium is unpolarised it means that \( \rho_{0,0}^0, \rho_{1,1}^1 \) and \( \rho_{1,1}^1 \) have to be equal and therefore one will expect to measure \( \lambda_\theta = 0.0 \), i.e. the decay products distribution does not
depend on \( \cos^2 \theta \). These exact same correspondences between polarisation states \( \lambda \) and polarisation observables have been found in [39].

While theoretically the knowledge of the parameters \( \lambda_\theta, \lambda_\phi, \) and \( \lambda_{\theta \phi} \) in a given polarisation frame of reference would give a complete description of the decay distribution, the experimental reality is that the measurement in a single frame could lead in certain cases to difficulties in the fitting of the angular distribution (4.26) to experimental data, due to non-ideal alignment of the polarisation axis. One will thus perform the theoretical predictions and measurements of the observables in several independent reference frames, in order to perform a cross-check.

D. \( \chi^J_c \) polarisation observables

As developed above the polarisation of the \( \chi^J_c \) meson is conveniently analysed experimentally by measuring the angular distribution of its decay (\( J/\psi \) in our study), which is parametrised using the three polarisation observables \( \lambda_\theta, \lambda_\phi, \) and \( \lambda_{\theta \phi} \), as:

\[
W^{J=1}(\theta, \phi) \propto 1 + \lambda_\theta \cos^2 \theta + \lambda_\phi \sin^2 \theta \cos(2\phi) + \lambda_{\theta \phi} \sin(2\theta) \cos \phi
\]

\[
W^{J=2}(\theta, \phi) \propto 1 + \lambda_\theta^{(1)} \cos^2 \theta + \lambda_\phi^{(2)} \cos^4 \theta + \lambda_\phi^{(1)} \sin^2 \theta \cos(2\phi) +
\]

\[
\quad + \lambda_\phi^{(2)} \sin^4 \theta \cos(2\phi) + \lambda_\phi^{(3)} \sin^4 \theta \cos(4\phi) + \lambda_{\theta \phi} \sin(2\theta) \cos \phi +
\]

\[
\quad + \lambda_{\theta \phi}^{(3)} \sin^2 \theta \sin(2\theta) \cos \phi
\]

where \( \theta \) and \( \phi \) are respectively the polar and the azimuthal angles of \( J/\psi \) in the \( \chi^J_c \) rest frame, as illustrated in the figure 24. The denition of these angles depends of course on the choice of coordinate frame. For \( J = 1 \), we rewrite (4.14) by comparison with (4.29) to find the \( \lambda \) coefficients:

\[
W^{J=1}(\theta, \phi) = \frac{3}{4\pi} \left[ 1 + \frac{r_1^1}{2} \lambda_\theta^{(1)} \right] \left[ 1 + \frac{r_1^0}{2} \right] \frac{\rho_{1,1}^1 - \rho_{0,0}^1}{(\rho_{0,0}^1 + \frac{r_1^1}{2}) \rho_{1,1}^1 + \frac{r_1^1}{2} \rho_{0,0}^1} \cos^2 \theta +
\]

\[
\quad + \frac{r_1^1}{2} - \frac{r_1^0}{2} \frac{\text{Re}(\rho_{1,-1}^1)}{(\rho_{0,0}^1 + \frac{r_1^1}{2}) \rho_{1,1}^1 + \frac{r_1^1}{2} \rho_{0,0}^1} \sin^2 \theta \cos(2\phi) +
\]

\[
\quad + \frac{r_1^1}{2} - \frac{r_1^0}{2} \frac{\sqrt{2} \text{Re}(\rho_{1,0}^1)}{(\rho_{0,0}^1 + \frac{r_1^1}{2}) \rho_{1,1}^1 + \frac{r_1^1}{2} \rho_{0,0}^1} \sin(2\theta) \cos \phi + \text{ other negligible terms in } \sin \phi
\]

which gives, with the numerical values of the r numbers, the polarisation coefficients for \( J = 1 \):

\[
\lambda_\theta^{J=1} = -\frac{\rho_{1,1}^1 - \rho_{0,0}^1}{3 \rho_{1,1}^1 + \rho_{0,0}^1}, \quad \lambda_\phi^{J=1} = \frac{\text{Re}(\rho_{1,-1}^1)}{3 \rho_{1,1}^1 + \rho_{0,0}^1}, \quad \lambda_{\theta \phi}^{J=1} = -\frac{\sqrt{2} \text{Re}(\rho_{1,0}^1)}{3 \rho_{1,1}^1 + \rho_{0,0}^1}
\]
For $J = 2$, we rewrite (4.15) to find the $\lambda$ coefficients:

\[
W^{J=2}(\theta, \phi) = \frac{5}{4\pi} \Delta \left[ 1 + \right.
\left. \frac{1}{\Delta} \left( 3(-r_0^2 + r_1^2 - \frac{r_0^2}{2})\rho_{0,0} + 3(2r_0^2 - r_1^2)\rho_{1,1}^2 + 3(-r_0^2 + \frac{r_1^2}{2})\rho_{2,2}^2 \right) \cos^2 \theta + \right.
\left. \frac{1}{\Delta} \left( 3(\frac{3r_0^2}{2} - r_1^2 + \frac{r_0^2}{4})\rho_{0,0} - (6r_0^2 - 4r_1^2 + r_2^2)\rho_{1,1}^2 + (\frac{3r_0^2}{2} - r_1^2 + \frac{r_2^2}{4})\rho_{2,2}^2 \right) \cos^4 \theta + \right.
\left. \frac{1}{\Delta} \left( 3(-2r_0^2 + r_1^2)\text{Re}(\rho_{1,-1}) + \sqrt{6}(2r_0^2 - 2r_1^2 + r_2^2)\text{Re}(\rho_{2,0}) \right) \sin^2 \theta \cos(2\phi) + \right.
\left. \frac{1}{\Delta} \left( (6r_0^2 - 4r_1^2 + r_2^2)\text{Re}(\rho_{1,-1}) - \frac{\sqrt{6}}{2} (6r_0^2 - 4r_1^2 + r_2^2)\text{Re}(\rho_{2,0}) \right) \sin^4 \theta \cos(2\phi) + \right.
\left. \frac{1}{\Delta} \left( \frac{1}{4} (6r_0^2 - 4r_1^2 + r_2^2)\text{Re}(\rho_{2,-2}) \right) \sin^4 \theta \cos(4\phi) + \right.
\left. \frac{1}{\Delta} \left( \sqrt{6}(-2r_0^2 + r_1^2)\text{Re}(\rho_{1,0}) - 2(r_1^2 - r_2^2)\text{Re}(\rho_{2,1}) \right) \sin(2\theta) \cos \phi + \right.
\left. \frac{1}{\Delta} \left( \frac{\sqrt{6}}{2} (6r_0^2 - 4r_1^2 + r_2^2)\text{Re}(\rho_{1,0}) - \frac{1}{2} (6r_0^2 - 4r_1^2 + r_2^2)\text{Re}(\rho_{2,1}) \right) \sin^2 \theta \sin(2\theta) \cos \phi + \right.
\left. \frac{1}{\Delta} \left( \frac{1}{2} (6r_0^2 - 4r_1^2 + r_2^2)\text{Re}(\rho_{2,-1}) \right) \sin^2 \theta \sin(2\theta) \cos(3\phi) + \text{other negligible terms in } \sin \phi \right]
\]

where

\[
\Delta = \left( \frac{r_0^2}{2} + \frac{3r_0^2}{4} \right)\rho_{0,0}^2 + \left( r_1^2 + r_2^2 \right)\rho_{1,1}^2 + \left( \frac{3r_0^2}{2} + r_1^2 + \frac{r_2^2}{4} \right)\rho_{2,2}^2
\]

which gives, with the numerical values of the $r$ numbers for E1 transitions (4.17), the polarisation coefficients for $J = 2$:

\[
\lambda^{J=2}_\theta \equiv \lambda^{(1)}_\theta = \frac{-3\rho_{0,0}^2 - 3\rho_{1,1}^2 + 6\rho_{2,2}^2}{5\rho_{0,0}^2 + 9\rho_{1,1}^2 + 6\rho_{2,2}^2}, \quad \lambda^{(2)}_\theta = 0
\]

\[
\lambda^{J=2}_\phi \equiv \lambda^{(1)}_\phi = \frac{3\text{Re}(\rho_{1,-1}) + 2\sqrt{6}\text{Re}(\rho_{2,0})}{5\rho_{0,0}^2 + 9\rho_{1,1}^2 + 6\rho_{2,2}^2}, \quad \lambda^{(2)}_\phi = 0, \quad \lambda^{(3)}_\phi = 0
\]

\[
\lambda^{J=2}_{\theta\phi} \equiv \lambda^{(1)}_{\theta\phi} = \frac{\sqrt{6}\text{Re}(\rho_{1,0}) + 6\text{Re}(\rho_{2,1})}{5\rho_{0,0}^2 + 9\rho_{1,1}^2 + 6\rho_{2,2}^2}, \quad \lambda^{(2)}_{\theta\phi} = 0, \quad \lambda^{(3)}_{\theta\phi} = 0
\]

As we do not analyse experimentally separately the contributions from $J=1$ and $J=2$, one need to find the polarisation observables for $\chi_c$:

\[
\frac{d\sigma^{\text{tot}}_{J/\psi} \text{ from } \chi_c}{d\Omega} \propto 1 + \lambda^{\text{tot}}_\theta \cos^2 \theta + \lambda^{\text{tot}}_\phi \sin^2 \theta \cos(2\phi) + \lambda^{\text{tot}}_{\theta\phi} \sin(2\theta) \cos \phi
\]

One can also write that the total cross section for $J/\psi$ from $\chi_c$ is the sum of the $J = 1$
and $J = 2$ contributions:

$$
\frac{d\sigma^{tot}_{J/\psi}}{d\Omega} = \frac{d\sigma^{J=1}_{J/\psi}}{d\Omega} + \frac{d\sigma^{J=2}_{J/\psi}}{d\Omega} = B(\chi^J_{c} \rightarrow J/\psi\gamma) \cdot \sigma^{J=1}_{\chi_c} \cdot W^{J=1}(\theta, \phi) + B(\chi^J_{c} \rightarrow J/\psi\gamma) \cdot \sigma^{J=2}_{\chi_c} \cdot W^{J=2}(\theta, \phi)
$$

4.36

$$
= c_1 \left[1 + \lambda^{J=1}_{\theta} \cos^2 \theta + \lambda^{J=1}_{\phi} \sin^2 \theta \cos(2\phi) + \lambda^{J=1}_{\theta\phi} \sin(2\theta) \cos \phi \right] + c_2 \left[1 + \lambda^{J=2}_{\theta} \cos^2 \theta + \lambda^{J=2}_{\phi} \sin^2 \theta \cos(2\phi) + \lambda^{J=2}_{\theta\phi} \sin(2\theta) \cos \phi \right]
$$

(c_1 + c_2) \left[1 + \left( \frac{c_1}{c_1 + c_2} \lambda^{J=1}_{\theta} + \frac{c_2}{c_1 + c_2} \lambda^{J=2}_{\theta} \right) \cos^2 \theta + \left( \frac{c_1}{c_1 + c_2} \lambda^{J=1}_{\phi} + \frac{c_2}{c_1 + c_2} \lambda^{J=2}_{\phi} \right) \sin^2 \theta \cos(2\phi) + \left( \frac{c_1}{c_1 + c_2} \lambda^{J=1}_{\theta\phi} + \frac{c_2}{c_1 + c_2} \lambda^{J=2}_{\theta\phi} \right) \sin(2\theta) \cos \phi \right]

$$
\text{where with (4.30) and (4.32):}
$$
c_1 = \frac{3}{4\pi} B(\chi^J_{c} \rightarrow J/\psi\gamma) \cdot \sigma^{J=1}_{\chi_c} \left( \rho^{1}_{1,0,0} + \rho^{1}_{1,1,1} \right) \left( \frac{r^{1}_{1,0,0}}{2} + \rho^{1}_{1,1,1} \left( \frac{r^{1}_{1,0,0}}{2} + \rho^{1}_{1,1,1} \right) \right)
$$

4.37

$$
c_2 = \frac{5}{8\pi} B(\chi^J_{c} \rightarrow J/\psi\gamma) \cdot \sigma^{J=2}_{\chi_c} \left( \rho^{2}_{2,0,0} + \rho^{2}_{2,1,1} \left( \frac{3\rho^{2}_{2,0,0}}{4} + 3\rho^{2}_{2,1,1} \right) \rho^{2}_{2,2,2} \right)
$$

4.38

With the numerical values of the $r$ coefficients for E1 transitions, it becomes:

$$
c_1 = \frac{3}{16\pi} B(\chi^J_{c} \rightarrow J/\psi\gamma) \left( \sigma^{1}_{1,0,0,0} + 3\sigma^{1}_{1,1,1,1} \right) \left( \frac{r^{1}_{1,0,0}}{2} + \sigma^{1}_{1,1,1} \left( \frac{r^{1}_{1,0,0}}{2} + \rho^{1}_{1,1,1} \right) \right)
$$

4.39

$$
c_2 = \frac{1}{16\pi} B(\chi^J_{c} \rightarrow J/\psi\gamma) \left( 5\sigma^{2}_{2,0,0,0} + 9\sigma^{2}_{2,1,1,1} + 6\sigma^{2}_{2,2,2,2} \right)
$$

Therefore with (4.35):

$$
\lambda^{tot}_{\theta} = \frac{c_1}{c_1 + c_2} \lambda^{J=1}_{\theta} + \frac{c_2}{c_1 + c_2} \lambda^{J=2}_{\theta}
$$

$$
\lambda^{tot}_{\phi} = \frac{c_1}{c_1 + c_2} \lambda^{J=1}_{\phi} + \frac{c_2}{c_1 + c_2} \lambda^{J=2}_{\phi}
$$

$$
\lambda^{tot}_{\theta\phi} = \frac{c_1}{c_1 + c_2} \lambda^{J=1}_{\theta\phi} + \frac{c_2}{c_1 + c_2} \lambda^{J=2}_{\theta\phi}
$$

Before giving the results for $\chi_c$ polarisation, we will study in the next section the polarisation of $J/\psi$ from $\chi_c$. 

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V. **J/ψ POLARISATION FROM ANGULAR DISTRIBUTION OF MUON PAIR**

A. **General scheme**

In the previous section we have studied $\chi_c$ polarisation by the study of the angular distribution of one of its decay products, $J/\psi$ in our case. Now if one wants to study $J/\psi$ polarisation, one need in the same way to study the angular distribution of one of its leptonic decay products (muonic in our study), being arbitrarily taken to be the $\mu^+$ muon. The new Feynman scheme for this study is therefore:

![Diagram](image)

**FIG. 28:** The double decay $\chi_c^\pm \rightarrow J/\psi \gamma \rightarrow \mu^+\mu^-$ process with the particle polarisations (in red) and their symbolic angles (in blue).

The polar angles $\Theta$ and $\Phi$ are the polar and azimuthal angles of the $\mu^+$ muon in the $J/\psi$ rest frame. These angles can be illustrated as in the figure 24, by replacing $\chi_c$ by $J/\psi$ and $J/\psi$ by $\mu^+$, and are defined from the production plane ($x,z$), i.e. the plane with colliding hadrons and $\chi_c^\pm$ momentum.

The decay of $J/\psi$ into two muons is mainly done by the intermediate of a virtual photon, indeed the muons do not possess SU(3) colors so they can not interact via the strong interaction (gluons), and as the energy of the non-relativistic $J/\psi$ is of the order of the GeV there is not enough energy involve (or it is really unlikely) to have intermediate weak bosons. As the photon is virtual it can transmit all possible $J/\psi$ polarisations, and therefore we will not consider it in the following developments.

Using the theory of decay into two spinning particles seen in Section IV A, one can write the $\mu^+$ production differential cross section from $J/\psi$ decay as:

\[
\frac{d\sigma_{\mu^+}}{d\Omega} = B(J/\psi \rightarrow \mu^+\mu^-) \cdot \sigma_{J/\psi} \cdot \tilde{W}^J(\Theta, \Phi),
\]

where \( \tilde{W}^J(\Theta, \Phi) = \sum_{\lambda_1,\lambda'_1=-J_1}^{J_1} \tilde{\rho}^J_{\lambda_1\lambda'_1} \tilde{A}^J_{\lambda_1\lambda'_1}(\Theta, \Phi) \) (5.1)
where \( B(J/\psi \rightarrow \mu^+\mu^-) = 0.0593 \) is the branching ratio of the corresponding decay, \( J_1 = 1 \) the total angular momentum of \( J/\psi \), \( d\Omega = \sin \Theta d\Theta d\Phi \) the solid angle element and \( \tilde{A}_{\lambda_1 \lambda'_1}(\Theta, \Phi) \) the decay angular distribution. Similarly to (4.3), the \( J/\psi \) polarised integrated hermitian density matrix for this decay is:

\[
\tilde{\rho}^J_{\lambda_1 \lambda'_1} \equiv \frac{\sigma^J_{\lambda_1 \lambda'_1}(J/\psi)}{\sigma^J_{J/\psi}}, \quad \text{where} \quad \sigma^J_{J/\psi} = \sum_{\lambda_1 = -J_1}^{J_1} \sigma^J_{\lambda_1 \lambda_1}(J/\psi) \quad (5.2)
\]

To compute (5.1) and its \( \mu^+ \) angular distribution \( \tilde{W}^J(\Theta, \Phi) \) (in fact the latter is the only thing that interests us in the frame of polarisation study), one first need to compute the \( J/\psi \) polarised density matrix \( \tilde{\rho}^J_{\lambda_1 \lambda'_1} \) and therefore \( \sigma^J_{\lambda_1 \lambda'_1}(J/\psi) \) (in (5.2)) the \( J/\psi \) polarised cross section for the \((pp \rightarrow \chi_c \rightarrow J/\psi \gamma)\) process, by integrating the following \( J/\psi \) polarised angular distribution \( W_{\lambda_1 \lambda'_1}(\theta, \phi) \) over the solid angle:

\[
\sigma^J_{\lambda_1 \lambda'_1}(J/\psi) = B(\chi_c \rightarrow J/\psi \gamma) \cdot \sigma^J_{\chi_c} \cdot \int W^J_{\lambda_1 \lambda'_1}(\theta, \phi) d\Omega \quad (5.3)
\]

where \( W^J_{\lambda_1 \lambda'_1}(\theta, \phi) = \sum_{\lambda, \lambda' = -J}^{J} \rho^J_{\lambda \lambda'_1} A^J_{\lambda \lambda'; \lambda_1 \lambda'_1}(\theta, \phi) \quad (5.4)\)

The latter expression is similar to (4.1) but without summing over \( \lambda_1 \) to keep the \( J/\psi \) polarisation information.

**B. \( J/\psi \) polarised integrated hermitian density matrix**

In the previous sections IV A. and IV B. we have found the angular distribution of the \( J/\psi \) unpolarised production from \( \chi_c^J \), because we have summed over \( J/\psi \) polarisation \( \lambda_1 \) in (4.4). In the polarised production case the decay angular distribution therefore becomes, the normalization factor remaining the same:

\[
A^J_{\lambda \lambda', \lambda_1 \lambda'_1}(\theta, \phi) = \frac{\sum_{\lambda_2} T^{J\ast}_{\lambda_2 \lambda', \lambda_1 \lambda'_1}(\theta, \phi) \cdot T^J_{\lambda_2 \lambda_1 \lambda_2}(\theta, \phi)}{\int \sum_{\lambda_1, \lambda_2} |T^J_{\lambda_1 \lambda_1 \lambda_2}(\theta, \phi)|^2 d\Omega} \quad (5.5)
\]

1. \( J = 1 \) case

Using the latter decay angular distribution and (5.3), we get the angular distributions of the \( J/\psi(\lambda_1 = -1, 0, 1) \) production from \( \chi_c^{J=1} \) with Mathematica:
The diagonal terms:

\[ W_{J=1}^{J=1}(\theta, \phi) = \frac{3}{8\pi} r_0^1 \left( \rho^1_{0,0} \cos^2 \theta + \rho^1_{1,1} \sin^2 \theta - \sqrt{2} \sin(2\theta) \left( \text{Re}(\rho^1_{1,0}) \cos \phi - \text{Im}(\rho^1_{1,0}) \sin \phi \right) - \sin^2 \theta \left( \text{Re}(\rho^1_{1,-1}) \cos(2\phi) - \text{Im}(\rho^1_{1,-1}) \sin(2\phi) \right) \right) \]  
\[ W_{J=0}^{J=0}(\theta, \phi) = \frac{3}{8\pi} r_1^1 \left( \rho^1_{0,0} \sin^2 \theta + \rho^1_{1,1} (1 + \cos^2 \theta) + \sqrt{2} \sin(2\theta) \left( \text{Re}(\rho^1_{1,0}) \cos \phi - \text{Im}(\rho^1_{1,0}) \sin \phi \right) + \sin^2 \theta \left( \text{Re}(\rho^1_{1,-1}) \cos(2\phi) - \text{Im}(\rho^1_{1,-1}) \sin(2\phi) \right) \right) \]  
\[ W_{J=1}^{J=-1}(\theta, \phi) = W_{J=1}^{J=1}(\theta, \phi) \]  
\[ W_{J=1}^{J=-1}(\theta, \phi) = \frac{3}{8\pi} \sqrt{r_0^1 r_1^1} \left( \rho^1_{0,0} - \rho^1_{1,1} \right) \sin(2\theta) + 2\sqrt{2} \cos(2\theta) \left( \text{Re}(\rho^1_{1,0}) \cos \phi - \text{Im}(\rho^1_{1,0}) \sin \phi \right) + \sin(2\theta) \left( \text{Re}(\rho^1_{1,-1}) \cos(2\phi) - \text{Im}(\rho^1_{1,-1}) \sin(2\phi) \right) \]  
\[ W_{J=0}^{J=-1}(\theta, \phi) = -W_{J=1}^{J=1}(\theta, \phi) = -W_{J=1}^{J=0}(\theta, \phi) = -W_{J=1}^{J=-1}(\theta, \phi) \]

These relations are logical with the properties (4.3) of the integrated hermitian density matrix.

Similarly to (4.30), the muonic angular distribution \( \tilde{W}_{J=1}(\Theta, \Phi) \), that will be computed in Section VC, requires only the (\( \lambda_1 = 0, \lambda'_1 = 0 \)), (\( \lambda_1 = 1, \lambda'_1 = 1 \)), (\( \lambda_1 = 1, \lambda'_1 = 0 \)) and (\( \lambda_1 = 1, \lambda'_1 = -1 \)) contributions. The relations (5.2), (5.3) and the integration of the \( W_{J=1}^{J=1}(\theta, \phi) \) calculated above over the solid angle, lead to the \( J/\psi \) polarised integrated hermitian density elements \( \tilde{\rho}_{J=1}^{(J=1)} \) function of the \( \chi_c \) integrated hermitian density elements \( \rho_{J=1}^{(J=1)} \):

\[ \tilde{\rho}_{0,0}^1 = \frac{1}{2} \left( \rho_{0,0}^1 + 2\rho_{1,1}^1 \right) \]  
\[ \tilde{\rho}_{0,0}^1 = \frac{1}{2} \rho_{0,0}^1 \]  
\[ \tilde{\rho}_{1,1}^1 = \frac{1}{2} \rho_{0,0}^1 \]  
\[ \tilde{\rho}_{1,-1}^1 = -\frac{1}{2} \rho_{0,0}^1 \]  
\[ \tilde{\rho}_{1,0}^1 = 0 \]

2. \( J = 2 \) case

The polarised angular distributions of the \( J/\psi(\lambda_1 = -1, 0, 1) \) production from \( \chi_c^{J=2} \) are:
The diagonal terms:

\[
W_{J=2}^{J=2} = \frac{5\sigma_{J=2}}{8\pi} \left\{ \frac{r_0^2}{2} (3 \cos^2 \theta - 1)^2 + \frac{3r_1^2}{4} \sin^4 \theta \right\} + \frac{\rho_1^2}{6} \left[ 6r_0^2 \sin^2 \theta \cos^2 \theta + r_2^2 \sin^2 \theta (\cos^2 \theta + 1) \right] + \frac{\rho_2^2}{2} \left[ 3r_0^2 \sin^4 \theta + \frac{r_2^2}{4} (\cos^4 \theta + 6 \cos^2 \theta + 1) \right] + \frac{\sqrt{6}}{2} \sin(2\theta) (2r_0^2 (1 - 3 \cos^2 \theta) + r_2^2 \sin^2 \theta) \left[ \text{Re}(\rho_1^2 \cos \phi) - \text{Im}(\rho_1^2 \sin \phi) \right] - \sin^2 \theta (6r_0^2 \cos^2 \theta - r_2^2 \sin^2 \theta) \left[ \text{Re}(\rho_1^2 \cos \phi) - \text{Im}(\rho_1^2 \sin \phi) \right] + \sin^3 \theta (6r_0^2 + r_2^2) \cos \theta \left( \text{Re}(\rho_2^2 \cos \phi) - \text{Im}(\rho_2^2 \sin \phi) \right) + \frac{1}{4} \sin \theta \left( \text{Re}(\rho_2^2 \cos \phi) - \text{Im}(\rho_2^2 \sin \phi) \right)
\]

\[
W_{J=2}^{J=2} = \frac{5\sigma_{J=2}}{8\pi} \left\{ \frac{3}{2} r_1^2 \rho_{0,0} \sin^2 \theta \cos^2 \theta + r_1^2 \rho_{1,1} (4 \cos^4 \theta - 3 \cos^2 \theta + 1) + \frac{r_2^2}{2} \rho_{2,2} \sin^2 \theta (\cos^2 \theta + 1) + \frac{\sqrt{6}}{2} r_1^2 \sin(2\theta) \cos(2\theta) \left[ \text{Re}(\rho_1^2 \cos \phi) - \text{Im}(\rho_1^2 \sin \phi) \right] - \frac{r_2^2}{2} \sin^2 \theta (\sin^2 \theta - 3 \cos^2 \theta) \left[ \text{Re}(\rho_1^2 \cos \phi) - \text{Im}(\rho_1^2 \sin \phi) \right] + 2r_1^2 \sin^3 \theta \left[ \cos \theta \left( \text{Re}(\rho_1^2 \cos \phi) - \text{Im}(\rho_1^2 \sin \phi) \right) + \frac{1}{4} \sin \theta \left( \text{Re}(\rho_2^2 \cos \phi) - \text{Im}(\rho_2^2 \sin \phi) \right) \right] + \sqrt{6} r_1^2 \sin^2 \theta \cos^2 \theta \left[ \text{Re}(\rho_2^2 \cos \phi) - \text{Im}(\rho_2^2 \sin \phi) \right] - r_2^2 \sin^2 \theta \cos \phi \left( \text{Re}(\rho_2^2 \cos \phi) - \text{Im}(\rho_2^2 \sin \phi) \right) \right\}
\]

\[
W_{J=2}^{J=2} = W_{J=2}^{J=2} (\theta, \phi) = W_{J=2}^{J=2} (\theta, \phi)
\]

The off-diagonal terms:

\[
W_{J=2}^{J=2} (\theta, \phi) = 0
\]

\[
W_{J=2}^{J=2} (\theta, \phi) = 0
\]
The expressions of $W_{\lambda_1=0, \lambda_1'=0}(\theta, \phi)$, $W_{\lambda_1=0, \lambda_1'=1}(\theta, \phi)$, $W_{\lambda_1=1, \lambda_1'=0}(\theta, \phi)$ and $W_{\lambda_1=1, \lambda_1'=0}(\theta, \phi)$ are too long to be shown in this text.

As for $J = 1$, the muonic angular distribution requires only the $(\lambda_1 = 0, \lambda_1' = 0)$, $(\lambda_1 = 1, \lambda_1' = 1)$, $(\lambda_1 = 1, \lambda_1' = 0)$ and $(\lambda_1 = 1, \lambda_1' = -1)$ contributions. The integration of the $W_{\lambda_1, \lambda_1'}(\theta, \phi)$ calculated above, over the solid angle, and with the angular cross section decomposition (5.3) and the second decay integrated hermitian density matrix (5.2), give the $J/\psi$ polarised integrated hermitian density elements $\tilde{\rho}_{\lambda,\lambda'}^{J=2}$ function of the $\chi_c$ integrated hermitian density elements $\rho_{\lambda,\lambda'}^{J=2}$:

\begin{align}
\tilde{\rho}_{0,0}^2 &= \frac{3}{5} \left( \frac{1}{2} \rho_{0,0}^2 + 2 \rho_{1,1}^2 + \rho_{2,2}^2 \right) \\
\tilde{\rho}_{1,1}^2 &= \frac{7}{5} \left( \frac{1}{2} \rho_{0,0}^2 + \rho_{1,1}^2 + \rho_{2,2}^2 \right) \\
\tilde{\rho}_{-1,-1}^2 &= 0 \\
\tilde{\rho}_{1,0}^2 &= 0
\end{align}

(5.11)

C. Muonic angular distribution

In the previous subsection we have computed the required density matrix elements $\tilde{\rho}_{\lambda,\lambda'}^{J=2}$ (corresponding to the polarised process $pp \to \chi_c \to J/\psi \gamma$), we shall derive the $\mu^+$ angular distribution $\tilde{W}_J(\Theta, \Phi)$ (corresponding to the process $J/\psi \to \mu^+ \mu^-$) following the same reasoning than in the "Decay into two particles" Section IV A.

Similarly to (5.12) and with $d\tilde{\Omega} = \sin \Theta d\Theta d\Phi$ the solid angle, the muonic decay angular distribution is given by the normalized product of the decay transition matrix $\tilde{T}$:

\begin{equation}
\tilde{A}_{\lambda_1, \lambda_1'}^{J=1}(\Theta, \Phi) = \frac{\sum \lambda_{\mu^+}, \lambda_{\mu^-} \tilde{T}_{\lambda_1, \lambda_1', \lambda_{\mu^+}, \lambda_{\mu^-}}^{J=1}(\Theta, \Phi) \cdot \tilde{T}_{\lambda_1, \lambda_1', \lambda_{\mu^+}, \lambda_{\mu^-}}^{J=1}(\Theta, \Phi)}{\int \sum \lambda_{\mu^+}, \lambda_{\mu^-} \tilde{T}_{\lambda_1, \lambda_1', \lambda_{\mu^+}, \lambda_{\mu^-}}^{J=1}(\Theta, \Phi)^2 d\tilde{\Omega}}
\end{equation}

(5.12)

where the transition matrix element $\tilde{T}_{\lambda_1, \lambda_1', \lambda_{\mu^+}, \lambda_{\mu^-}}^{J=1}(\Theta, \Phi)$ of the muonic decay process $J/\psi \to \mu^+(\Theta, \Phi) + \mu^-(\pi - \Theta, \Phi + \pi)$, in the considered frame of reference, can be decomposed into a decay amplitude $T_{\lambda_1, \lambda_1', \lambda_{\mu^+}, \lambda_{\mu^-}}^{J=1}$ and a representation of the rotation group $D$ that describes the change in quantization orientations from the state $|J_1, \lambda_1\rangle$ to $|\lambda_{\mu^+}, \lambda_{\mu^-}, \Theta, \Phi\rangle$:

\begin{align}
\tilde{T}_{\lambda_1, \lambda_1', \lambda_{\mu^+}, \lambda_{\mu^-}}^{J=1}(\Theta, \Phi) &= \langle \lambda_{\mu^+}, \lambda_{\mu^-}, \Theta, \Phi | \tilde{T} | J_1, \lambda_1 \rangle = \sqrt{\frac{2J_1 + 1}{4\pi}} T_{\lambda_1, \lambda_1', \lambda_{\mu^+}, \lambda_{\mu^-}}^{J=1} D_{\lambda_1, \lambda_1', \lambda_{\mu^+}, \lambda_{\mu^-}}^{J=1}(-\Phi, \Theta, \Phi), \\
\tilde{T}_{\lambda_1, \lambda_1', \lambda_{\mu^+}, \lambda_{\mu^-}}^{J=1} &= \sqrt{\frac{4\pi}{2J_1 + 1}} \langle \lambda_{\mu^+}, \lambda_{\mu^-}, 0, 0 | \tilde{T} | J_1, \lambda_1 \rangle |_{\lambda_1=\lambda_{\mu^+}=-\lambda_{\mu^-}}
\end{align}

(5.13)

The parity (symmetry of the field through space inversion) of a particle is given by the products of its intrinsic parities (1 for a charged fermion and -1 for an charged antifermion) and of its extrinsic parity (equal to $(-1)^J$). Let $\eta_1 = -1$, $\eta_{\mu^+} = -1$ and $\eta_{\mu^-} = 1$ be
respectively the parities of \( J/\psi \), \( \mu^+ \) with \( J_{\mu^+} = S_{\mu^+} = 1/2 \) and \( \mu^- \) with \( J_{\mu^-} = S_{\mu^-} = 1/2 \). Due to parity conservation the decay amplitude \( T^{J_1}_{\lambda_{\mu^+}, 0} \) satisfies the following polarisation symmetry:

\[
\hat{T}^{J_1}_{\lambda_{\mu^+}, -\lambda_{\mu^-}} = \eta_1 \eta_{\mu^+} \eta_{\mu^-} (1 - J_{\mu^-}) \hat{T}^{J_1}_{\lambda_{\mu^+}, 0} = \hat{T}^{J_1}_{\lambda_{\mu^+}, 0}
\]  

(5.14)

The possible \( \mu^+ \) and \( \mu^- \) polarisation states are not only limited by the usual rules \( |\lambda_{\mu^+}| \leq J_{\mu^+} \) and \( |\lambda_{\mu^-}| \leq J_{\mu^-} \) but also by the angular momentum conservation that imposes \( |\lambda_{\mu^+} - \lambda_{\mu^-}| \leq J_1 \). Indeed as \( \mu^+ \) and \( \mu^- \) go in opposite directions, the polarisation of the system \( [\mu^+; \mu^-] \) is \( |\lambda_{\mu^+} - \lambda_{\mu^-}| \). Therefore, it follows by using (5.14) that we can define:

\[
l_{0}^{1} \equiv \hat{T}_{-\frac{1}{2}, -\frac{1}{2}}^{1} = T_{-\frac{1}{2}, -\frac{1}{2}}^{1}, \quad l_{1}^{1} \equiv \hat{T}_{\frac{1}{2}, -\frac{1}{2}}^{1} = T_{\frac{1}{2}, -\frac{1}{2}}^{1}
\]  

(5.15)

Combining with Mathematica, equation (5.1), all the relations above and some related to the \( D \) operator in Section IV A, we get the angular distribution of the \( \mu^+ \) production from \( J/\psi \):

\[
\hat{W}^{J}(\Theta, \Phi) = \frac{3}{4\pi} \left( \hat{r}_{0,0}^J \cos^2 \Theta + \frac{\hat{r}_{1,0}^J}{2} \sin^2 \Theta \right) + \hat{\rho}_{1,1}^J \left[ \hat{r}_{0,0}^J \sin^2 \Theta + \frac{\hat{r}_{1,0}^J}{2} (1 + \cos^2 \Theta) \right] - \sqrt{2} \sin(2\Theta) \left( \hat{r}_{0,0}^J - \frac{\hat{r}_{1,0}^J}{2} \right) \left[ \Re(\hat{\rho}_{1,0}^J) \cos \Phi - \Im(\hat{\rho}_{1,0}^J) \sin \Phi \right] - \sin^2 \Theta \left( \hat{r}_{0,0}^J - \frac{\hat{r}_{1,0}^J}{2} \right) \left[ \Re(\hat{\rho}_{1,-1}^J) \cos(2\Phi) - \Im(\hat{\rho}_{1,-1}^J) \sin(2\Phi) \right]
\]  

(5.16)

where

\[
\hat{r}_{\lambda^J}^J = \sum_{\lambda^J=0}^{J} \frac{|\hat{r}_{\lambda^J}^J|^2}{|\hat{r}_{\lambda^J}^J|^2}
\]  

(5.17)

By analogy with the process \( \chi^{J=-1}_c \rightarrow J/\psi \gamma \) where we had \( r_0^J \simeq r_1^J \simeq 0.5 \) from measurements (see (4.17)), i.e. there was an equiprobability that the couple \( J/\psi \gamma \) comes from \( \chi_c \) with \( \lambda = 1, -1 \) or \( \chi_c \) from \( \lambda = 0 \), one can reasonably assume that for our current process \( J/\psi \rightarrow \mu^+ \mu^- \) there is also this equiprobability that the couple \( \mu^+ \mu^- \) comes from \( J/\psi \) with \( \lambda^J = 1, -1 \) or \( \chi_c \) with \( \lambda = 0 \), i.e. \( r_0^J \simeq r_1^J \simeq 0.5 \). We emphasis that this is an assumption and that a experimental measurement would be required here.

The expression (5.16) of \( \hat{W}^{J}(\Theta, \Phi) \) turns out to be equivalent to the one (4.14) of \( W^{J}(\theta, \phi) \) despite the different parity, total angular momentum and polarisation values of the decay products.

D. Polariation observables for \( J/\psi \) from \( \chi_c \)

In Section IV D, the polarisation of the \( \chi^J_c \) meson was conveniently analysed experimentally by measuring the angular distribution of its decay (\( J/\psi \) in our study), which was
parametrized using the three polarisation observables $\lambda_\theta$, $\lambda_\phi$, and $\lambda_{\theta\phi}$. Similarly the polarisation of the $J/\psi$ meson is conveniently analysed experimentally by measuring the angular distribution of its decay ($\mu^+$ in our study), which is parametrized using the three polarisation observables $\lambda_\theta^J$, $\lambda_\phi^J$, and $\lambda_{\theta\phi}^J$

$$
\tilde{W}^J(\Theta, \Phi) \propto 1 + \lambda_\theta^J \cos^2 \Theta + \lambda_\phi^J \sin^2 \Theta \cos(2\Phi) + \lambda_{\theta\phi}^J \sin(2\Theta) \cos \Phi \quad (5.18)
$$

We rewrite (5.16) by comparison with (5.18) to find the $\lambda$ coefficients:

$$
\tilde{W}^J(\Theta, \Phi) = \frac{3}{4\pi} \left( \tilde{\rho}_{0,0}^J - \tilde{\rho}_{1,1}^J(\tilde{r}_0^1 + \tilde{r}_1^1) \right) \left[ 1 + \frac{\tilde{r}_0^1 - \tilde{r}_1^1}{\tilde{r}_0^1 + \tilde{r}_1^1} \frac{\tilde{\rho}_{1,1}^J - \tilde{\rho}_{0,0}^J}{\tilde{\rho}_{1,1}^J + \frac{\tilde{r}_1^1}{2} \tilde{\rho}_{0,0}^J} \cos^2 \Theta + \left( \frac{\tilde{r}_1^1}{2} - \tilde{r}_0^1 \right) \frac{\text{Re}(\tilde{\rho}_{1,1}^{J-})}{\tilde{r}_0^1 + \tilde{r}_1^1} \sin^2 \Theta \cos(2\Phi) + \left( \frac{\tilde{r}_1^1}{2} - \tilde{r}_0^1 \right) \frac{\sqrt{2} \text{Re}(\tilde{\rho}_{0,0}^{J})}{\tilde{r}_0^1 + \tilde{r}_1^1} \sin(2\Theta) \cos \Phi + \text{other negligible terms in } \sin \Phi \right] \quad (5.19)
$$

which gives, with the numerical values of the $\tilde{r}$ numbers, the polarisation coefficients:

$$
\lambda_\theta^J = - \frac{\tilde{\rho}_{1,1}^J - \tilde{\rho}_{0,0}^J}{3\tilde{\rho}_{1,1}^J + \tilde{\rho}_{0,0}^J}, \quad \lambda_\phi^J = - \frac{\text{Re}(\tilde{\rho}_{1,1}^{J-})}{3\tilde{\rho}_{1,1}^J + \tilde{\rho}_{0,0}^J}, \quad \lambda_{\theta\phi}^J = - \frac{\sqrt{2} \text{Re}(\tilde{\rho}_{0,0}^{J})}{3\tilde{\rho}_{1,1}^J + \tilde{\rho}_{0,0}^J} \quad (5.20)
$$

These polarisation coefficients can be rewritten in function of the density matrix elements of the $pp \to \chi_c(J = 1) \to J/\psi/\gamma$ process using (5.8), and that yields:

$$
\lambda_\theta^{J=1} = \frac{1}{5}, \quad \lambda_\phi^{J=1} = \frac{1}{5}, \quad \lambda_{\theta\phi}^{J=1} = 0 \quad (5.21)
$$

And for $J = 2$:

$$
\lambda_\theta^{J=2} = - \frac{2\tilde{\rho}_{0,0}^2 + \tilde{\rho}_{1,1}^2 + 4\tilde{\rho}_{2,2}^2}{9\tilde{\rho}_{0,0}^2 + 15\tilde{\rho}_{1,1}^2 + 18\tilde{\rho}_{2,2}^2}, \quad \lambda_\phi^{J=2} = 0, \quad \lambda_{\theta\phi}^{J=2} = 0 \quad (5.22)
$$

As experimentally one does not distinguish between the $J = 1$ and $J = 2$ contributions one needs to search for the total polarisation coefficients:

$$
\frac{d\sigma_{\mu^+}}{d\Omega} \propto 1 + \lambda_{\theta}^{J=tot} \cos^2 \Theta + \lambda_{\phi}^{J=tot} \sin^2 \Theta \cos(2\Phi) + \lambda_{\theta\phi}^{J=tot} \sin(2\Theta) \cos \Phi \quad (5.23)
$$

Using the same reasoning as we did for $\chi_c$ in (4.36):

$$
\frac{d\sigma_{J=tot}}{d\Omega} = \frac{d\sigma_{J=1}^{tot}}{d\Omega} + \frac{d\sigma_{J=2}^{tot}}{d\Omega} = \frac{3}{4\pi} \mathcal{B}(J/\psi \to \mu^+ \mu^-)(\tilde{c}_1 + \tilde{c}_2) \left[ 1 + \left( \frac{\tilde{c}_1}{\tilde{c}_1 + \tilde{c}_2} \lambda_{\theta}^{J=1} + \frac{\tilde{c}_2}{\tilde{c}_1 + \tilde{c}_2} \lambda_{\phi}^{J=2} \right) \cos^2 \Theta + \left( \frac{\tilde{c}_1}{\tilde{c}_1 + \tilde{c}_2} \lambda_{\theta}^{J=1} + \frac{\tilde{c}_2}{\tilde{c}_1 + \tilde{c}_2} \lambda_{\phi}^{J=2} \right) \sin^2 \Theta \cos(2\Phi) + \left( \frac{\tilde{c}_1}{\tilde{c}_1 + \tilde{c}_2} \lambda_{\theta\phi}^{J=1} + \frac{\tilde{c}_2}{\tilde{c}_1 + \tilde{c}_2} \lambda_{\theta\phi}^{J=2} \right) \sin(2\Theta) \cos \Phi \right] \quad (5.24)
$$

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where with (5.19):

\[
\tilde{c}_1 = B(\chi_c^{J=1} \rightarrow J/\psi\gamma) \cdot \sigma_{\chi_c}^{J=1} \cdot \left(\tilde{\rho}_{0,0}^{J=1} \frac{\tilde{r}_1^1}{2} + \tilde{\rho}_{1,1}^{J=1} (\tilde{r}_0^1 + \frac{\tilde{r}_1^1}{2})\right) \tag{5.25}
\]

\[
\tilde{c}_2 = B(\chi_c^{J=2} \rightarrow J/\psi\gamma) \cdot \sigma_{\chi_c}^{J=2} \cdot \left(\tilde{\rho}_{0,0}^{J=2} \frac{\tilde{r}_1^1}{2} + \tilde{\rho}_{1,1}^{J=2} (\tilde{r}_0^1 + \frac{\tilde{r}_1^1}{2})\right)
\]

in which we have used:

\[
\sigma_{J/\psi}^J = B(\chi_c^J \rightarrow J/\psi\gamma) \cdot \sigma_{\chi_c}^J \tag{5.26}
\]

With the numerical values of the \(\tilde{r}\) coefficients and the equations (5.8) and (5.11), the coefficients become:

\[
\tilde{c}_1 = \frac{5}{8} B(\chi_c^{J=1} \rightarrow J/\psi\gamma) \left(\frac{1}{2} \sigma_{0,0}^1 + \sigma_{1,1}^1\right) \tag{5.27}
\]

\[
\tilde{c}_2 = \frac{3}{20} B(\chi_c^{J=2} \rightarrow J/\psi\gamma) \left(4 \sigma_{0,0}^2 + 9 \sigma_{1,1}^2 + 8 \sigma_{2,2}^2\right)
\]

Therefore with (5.23) the total coefficients for \(J/\psi\) polarisation from \(\chi_c\) are:

\[
\lambda_{\Theta}^{tot} = \frac{\tilde{c}_1}{\tilde{c}_1 + \tilde{c}_2} \lambda_{\Theta}^{J=1} + \frac{\tilde{c}_2}{\tilde{c}_1 + \tilde{c}_2} \lambda_{\Theta}^{J=2} \tag{5.28}
\]

\[
\lambda_{\Phi}^{tot} = \frac{\tilde{c}_1}{\tilde{c}_1 + \tilde{c}_2} \lambda_{\Phi}^{J=1} \quad \lambda_{\Theta\Phi}^{tot} = 0
\]

The angular distribution of the muonic pair have been analytically derived and the polarisation parameters \(\lambda_{\Theta}^{tot}\), \(\lambda_{\Phi}^{tot}\) and \(\lambda_{\Theta\Phi}^{tot}\) are obtained. In the following section we analyse numerically the latter observables for \(J/\psi\) polarisation and the ones from the previous section for \(\chi_c\) polarisation.
VI. POLARISATION RESULTS FOR $\chi_c$ AND $J/\psi$ FROM $\chi_c$

A. Previous results for total $J/\psi$ polarisation

The Tevatron polarisation data [49] correspond to prompt $J/\psi$ production and LHC data [50] correspond to total $J/\psi$ [51] production i.e. prompt and non-prompt production. However previous works on $J/\psi$ polarisation predictions only consider direct $J/\psi$ production. Indeed, on one hand the polarisation of $J/\psi$ from $\psi'$ is identical to the one for direct $J/\psi$ as the scalar pions decay do not change the polarisation of the meson and on other hand the polarisation of $J/\psi$ from $\chi_c$ has been assumed to be negligible because of the radiative decay. The latter assumption comes from the fact that the random photon polarisation should take away in mean the polarisation of the meson. As you will see in the following sections, our results of the polarisation of $J/\psi$ from $\chi_c$ tend to prove their assumption.

One of the previous predictions, carried out by Butenschoen et al. [52] within the collinear factorisation approach for only direct $J/\psi$ production at leading order (LO) and next-to-leading order (NLO), is presented in fig. 29. As you can see there is a strong disagreement between experimental data and their predictions.

As in the frame of the present thesis we do not compute the predictions for direct $J/\psi$ we will not be able to compare directly their results, within the collinear factorisation approach, with ours within the $k_\perp$ factorisation one. However, in the following, we analyse our predictions for the polarisation observable dependences on the meson transverse momentum and rapidity for the meson $\chi_c^{J=1}$, $\chi_c^{J=2}$, $\chi_c$ and $J/\psi$ from $\chi_c$ for Tevatron and LHC energies.

![Fig. 29: Predictions of the $J/\psi$ polarisation within the collinear factorisation approach for both CS and CO models at LO and NLO, by Butenschoen et al. [52].](image)
B. Polarisation results for Tevatron Run I and II

In the following, we look at the transverse momentum $P_t$ dependence of the following differential polarisation observables in the Helicity frame (HX): $\lambda^J=1$ for $\chi^{J=1}_c$ (expression (4.31)), $\lambda^J=2$ for $\chi^{J=2}_c$ (expression (4.34)), $\lambda^{tot}$ for $\chi_c$ (expression (4.39)) and $\lambda^{tot}$ for $J/\psi$ from $\chi_c$ (expression (5.28)).

As the error UGDFs band has revealed to be really small (the errors cancel because of the ratios of cross sections), they are not depicted in the following figures. The rapidity $y$ is integrated over the interval [-4,4] and $|k_{1,t}|$ is integrated over the interval [0.6,40]. We emphasis once again that NLO means with the NLO factor applied to the $\chi^{J=2}_c$ production but as it has not been yet calculated not to the $\chi^{J=1}_c$ production.

FIG. 30: [HX frame, NLO, scale $\mu^2 = M^2_t/4$, pseudorapidity cut $|\eta| < 0.6$, $\sqrt{s} = 1.8$ TeV] The different polarisation observables $\lambda_\theta$ (see text) dependence on the meson transverse momentum for the UGDF CCFM set B0 in indirect comparison to the CDF data [49] for $J/\psi$ prompt production (direct, from $\chi_c$ and $\psi'$). Analysis: As explained in the polarisation observables introduction in Section IV C 2, $\lambda^{J=1}_\theta \rightarrow 1$ ($\lambda^{J=2}_\theta \rightarrow -0.6$) at large $P_t$ means that $\chi^{J=1}_c$ ($\chi^{J=2}_c$) has a fully longitudinal polarisation $\lambda = 0$ at large $P_t$. The radiative decay between $\chi_c$ and $J/\psi$ clearly changes the polarisation between the two meson states as the two corresponding curves are not identical. The $\lambda^{tot}_\theta$ for $J/\psi$ from $\chi_c$ is really close to zero for all $P_t$, which means that the $J/\psi$ from $\chi_c$ are unpolarised along the HX polarisation axis.
FIG. 31: [HX frame, LO, scale $\mu^2 = M_t^2/4$, pseudorapidity cut $|\eta| < 0.6$, $\sqrt{s} = 1.8$ TeV] Same as in fig. 30 but without the NLO factors. Analysis: The $\lambda_{\text{tot}}$ for $J/\psi$ from $\chi_c$ remains close to zero for all $P_t$. The $\lambda_{\theta}^{J=1}$ and $\lambda_{\theta}^{J=2}$ are identical to the ones with NLO factors, so only the coefficients $\tilde{c}_{1,2}$ ($c_{1,2}$) from (5.27) ((4.38)) make the changes to $J/\psi$ from $\chi_c$ ($\chi_c$).

FIG. 32: [NLO, scale $\mu^2 = M_t^2/4$, pseudorapidity cut $|\eta| < 0.6$, $\sqrt{s} = 1.8$ TeV] Influence of the frame choice on $\lambda_{\theta}^{\text{tot}}$ for $J/\psi$ from $\chi_c$ and on $\lambda_{\theta}^{\text{tot}}$ for $\chi_c$. Analysis: At this energy the Collins-Soper and Target frames give identical results and are really close to the Helicity frame results. For all frames, the $\lambda_{\theta}^{\text{tot}}$ being really close to 0, $J/\psi$ from $\chi_c$ is indeed unpolarised whatever is the direction of the polarisation axis.
FIG. 33: [HX frame, NLO, scale $\mu^2 = M_t^2/4$, pseudorapidity cut $|\eta| < 0.6$, $\sqrt{s} = 1.96$ TeV] Same as in fig. 30 but at Tevatron Run II energy (CDF data [50]). Analysis: The change of energy between the Run I and II does not make any difference in our predictions.

C. Polarisation results for LHC Run I and II

As you have already seen in fig. 32 and as you will see in fig. 36, the HX, T and GJ frames give (almost) identical results for LHC (Tevatron) energies. It can be easily explained by looking at the fig. 27: indeed, at these high collision energies, the produced meson has a relatively important momentum in the c.m.s. frame and therefore when the Lorentz boost is performed to go to its rest frame, the HX, T and GJ polarisation axis turn to be in the same direction (in other words, when the meson momentum is large the angle between the T (or GJ) and CS frames becomes almost 90).

The above predictions for the polarisation observables at the LHC at energy $\sqrt{s} = 7$ TeV turn to be the same as the ones at energy $\sqrt{s} = 14$ TeV. Indeed, from a certain colliding energy value these results do not depend on the latter as the polarisation frames do not change anymore and as the polarisation observables correspond to a ratio of cross sections. With the analysis we did just above, we can deduce that this colliding energy value from which the results do not change is in between Tevatron $\sqrt{s} = 1.96$ TeV and LHC $\sqrt{s} = 7$ TeV.

In the following, we look at the transverse momentum $p_T$ dependence of the following differential polarisation observables in the Helicity frame (HX) and Collin-Soper frame (CS): $\lambda_{j=1}$ for $\chi_{j=1}^J$ (expression (4.31)), $\lambda_{j=2}$ for $\chi_{j=2}^J$ (expression (4.34)), $\lambda^{tot}_{G}$ for $\chi_c$ (expression 4.39) and $\lambda^{tot}_G$ for $J/\psi$ from $\chi_c$ (expression (5.28)).
FIG. 34: [HX frame, NLO, scale $\mu^2 = M_t^2/4$, rapidity cut $2.5 < y < 4$, $\sqrt{s} = 7$ TeV] The different polarisation observables $\lambda_\theta$ (see text) dependence on the meson transverse momentum for the UGDF CCFM set B0 in indirect comparison to the ALICE data [51] for $J/\psi$ total production (prompt and non-prompt). Analysis: Same as in fig. 30.

FIG. 35: [CS frame, NLO, scale $\mu^2 = M_t^2/4$, rapidity cut $2.5 < y < 4$, $\sqrt{s} = 7$ TeV] The different polarisation observables $\lambda_\theta$ (see text) dependence on the meson transverse momentum for the UGDF CCFM set B0 in indirect comparison to the ALICE data [51] for $J/\psi$ total production (prompt and non-prompt). Analysis: As explained in the polarisation observables introduction in Section IV C 2, $\lambda_\theta^{J=1} = -1/3$ means that $\chi^{J=1}_c$ has a fully transverse polarisation $\lambda = \pm 1$, along the CS frame polarisation axis, for all $P_t$. Moreover, $\lambda_\theta^{J=2} \to 1$ at small $P_t$ means that $\chi^{J=2}_c$ has a fully transverse polarisation $\lambda = \pm 2$ at small $P_t$. The radiative decay between $\chi_c$ and $J/\psi$ changes the polarisation between the two meson states as the two corresponding curves are not identical. The $\lambda_\theta^{tot}$ for $J/\psi$ from $\chi_c$ is once again really close to zero for all $P_t$, which means that the $J/\psi$ from $\chi_c$ are unpolarised along the CS polarisation axis.
FIG. 36: [NLO, scale $\mu^2 = M_t^2/4$, rapidity cut $2.5 < y < 4$, $\sqrt{s} = 7$ TeV] Influence of the frame choice on $\lambda_{tot}^\Theta$ for $J/\psi$ from $\chi_c$ and on $\lambda_{tot}^\phi$ for $\chi_c$. Analysis: At this energy the Collins-Soper, Target and Helicity frames give identical results. For all frames, the $\lambda_{tot}^\Theta$ being really close to 0, $J/\psi$ from $\chi_c$ is indeed unpolarised whatever is the direction of the polarisation axis.

For the next two figures, we look at the transverse momentum $P_t$ dependence of the following differential polarisation observables in the HX and CS frames: $\lambda_{\phi}^{J=1}$ for $\chi_c^{J=1}$ (expression (4.31)), $\lambda_{\phi}^{J=2}$ for $\chi_c^{J=2}$ (expression (4.34)), $\lambda_{\phi}^{tot}$ for $\chi_c$ (expression (4.39)) and $\lambda_{\phi}^{tot}$ for $J/\psi$ from $\chi_c$ (expression (5.28)).

FIG. 37: [HX frame, NLO, scale $\mu^2 = M_t^2/4$, rapidity cut $2.5 < y < 4$, $\sqrt{s} = 7$ TeV] The different polarisation observables $\lambda_{\phi}$ (see text) dependence on the meson transverse momentum for the UGDF CCFM set B0 in indirect comparison to the ALICE data [51] for $J/\psi$ total production (prompt and non-prompt). Analysis: All the $\chi_c$ curves are close to zero and the $J/\psi$ from $\chi_c$ curve remains small ($\lambda_{\phi}^{tot} \rightarrow 0.1$) but not null as we have for $\lambda_{\phi}^{tot}$. Therefore, in the HX frame, the decay product angular distributions are almost independent of the angular term $\sin^2 \Theta \cos(2\Phi)$. 
FIG. 38: [CS frame, NLO, scale $\mu^2 = M_t^2/4$, rapidity cut $2.5 < y < 4$, $\sqrt{s} = 7$ TeV] The different polarisation observables $\lambda_{\phi}$ (see text) dependence on the meson transverse momentum for the UGDF CCFM set B0 in indirect comparison to the ALICE data [51] for $J/\psi$ total production (prompt and non-prompt). Analysis: In the CS frame, the $\chi_c$ and $J/\psi$ from $\chi_c$ curves are close and almost constantly small (equal to 0.1).

For the next four figures, we look at the rapidity dependence of the following differential polarisation observables in the HX and CS frames: $\lambda^{J=1}_{\theta}$ and $\lambda^{J=1}_{\phi}$ for $\chi_{J=1}^c$ (expression (4.31)), $\lambda^{J=2}_{\theta}$ and $\lambda^{J=2}_{\phi}$ for $\chi_{J=2}^c$ (expression (4.34)), $\lambda^{\text{tot}}_{\theta}$ and $\lambda^{\text{tot}}_{\phi}$ for $\chi_c$ (expression (4.39)) and $\lambda^{\text{tot}}_{\Theta}$ and $\lambda^{\text{tot}}_{\Phi}$ for $J/\psi$ from $\chi_c$ (expression (5.28)).

FIG. 39: [HX frame, NLO, scale $\mu^2 = M_t^2/4$, $P_t < 20$ GeV, $\sqrt{s} = 7$ TeV] The different polarisation observables $\lambda_{\theta}$ (see text) dependence on the meson rapidity for the UGDF CCFM set B0. Analysis: At large rapidity $\lambda^{J=1}_{\theta} \rightarrow 0$ which means that $\chi_{J=1}^c$ becomes unpolarised at large rapidity. The $\lambda^{\text{tot}}_{\Theta}$ for $J/\psi$ from $\chi_c$ is rapidity independent and almost null.
FIG. 40: [CS frame, NLO, scale $\mu^2 = M^2_t/4$, $P_t < 20$ GeV, $\sqrt{s} = 7$ TeV] The different polarisation observables $\lambda_\phi$ (see text) dependence on the meson rapidity for the UGDF CCFM set B0. Analysis: In the CS frame the $\lambda_\phi$ are almost rapidity independent. As we have $\lambda^{J=1}_\theta = -1/3$ it means that $\chi^{J=1}_c$ has a fully transverse polarisation $\lambda = \pm 1$ (which is consistent with the $P_t$ result in fig. 35). Moreover $\lambda^{J=2}_\theta \approx 1$ it means that $\chi^{J=2}_c$ has a fully transverse polarisation $\lambda = \pm 2$. The $\lambda^{tot}_\Theta$ for $J/\psi$ from $\chi_c$ is once again small but not null.

FIG. 41: [HX frame, NLO, scale $\mu^2 = M^2_t/4$, $P_t < 20$ GeV, $\sqrt{s} = 7$ TeV] The different polarisation observables $\lambda_\phi$ (see text) dependence on the meson rapidity for the UGDF CCFM set B0. Analysis: The $\lambda^{tot}_\Theta$ for $J/\psi$ from $\chi_c$ is rapidity independent and small.
FIG. 42: [CS frame, NLO, scale $\mu^2 = M_T^2/4$, $P_t < 20 \text{ GeV}$, $\sqrt{s} = 7 \text{ TeV}$] The different polarisation observables $\lambda_\phi$ (see text) dependence on the meson rapidity for the UGDF CCFM set B0. **Analysis:** Once again, in the CS frame the $\lambda_\phi$ are almost rapidity independent. The $\lambda^{tot}_\Phi$ for $J/\psi$ from $\chi_c$ is small but not null.

For the last two figures (they are also valid for LHC $\sqrt{s} = 7 \text{ TeV}$), we look at the transverse momentum $P_t$ dependence of the following differential polarisation observables in the Helicity frame (HX) and Collin-Soper frame (CS): $\lambda^{j=1}_{\theta\phi}$ for $\chi_c^{j=1}$ (expression (4.31)), $\lambda^{j=2}_{\theta\phi}$ for $\chi_c^{j=2}$ (expression (4.34)), $\lambda^{tot}_{\theta\phi}$ for $\chi_c$ (expression 4.39) and $\lambda^{tot}_{\Phi\Phi}$ for $J/\psi$ from $\chi_c$ (expression (5.28)).

FIG. 43: [HX frame, NLO, scale $\mu^2 = M_T^2/4$, rapidity cut $2.5 < y < 4$, $\sqrt{s} = 7 \text{ TeV}$] The different polarisation observables $\lambda_{\theta\phi}$ (see text) dependence on the meson transverse momentum for the UGDF CCFM set B0. **Analysis:** The observable $\lambda^{tot}_{\Phi\Phi}$ for $J/\psi$ from $\chi_c$ is not represented as we found that it is null (see (5.28)).
FIG. 44: [CS frame, NLO, scale $\mu^2 = M_t^2/4$, rapidity cut $2.5 < y < 4$, $\sqrt{s} = 7$ TeV] The different polarisation observables $\lambda_{\theta\phi}$ (see text) dependence on the meson transverse momentum for the UGDF CCFM set B0. *Analysis:* The observable $\lambda_{\theta\phi}^{tot}$ for $J/\psi$ from $\chi_c$ is not represented as we found that it is null (see (5.28)). In the CS frame all the $\lambda_{\theta\phi}$ observables turn to be null: the decay product angular distributions are almost independent of the angular term $\sin(2\Theta)\cos\Phi$.

These polarisation observables have therefore been calculated for the incident charmonia states $\chi_c^{J=1}$, $\chi_c^{J=2}$, $\chi_c$ (total) and $J/\psi$ from radiative decay of $\chi_c$. They have been analysed numerically in four different polarisation frames (Helicity, Collin-Soper, Gottfried-Jackson and Target frames). Results demonstrate especially the vanishing polarisation of $J/\psi$ coming from $\chi_c$ decays at all energies, thus confirming previous hypothesis declared in the literature. We have shown that proposed approach based on $k_T$ factorisation in NRQCD framework leads to results compatible with small or negligible $J/\psi$ polarisation which is consistent with the vanishing LHC data for total $J/\psi$ polarisation.
VII. CONCLUSION AND OUTLOOK

The original results of the Master thesis can be summarized as follows:

- For the first time, the $k_\perp$ factorisation approach in non relativistic QCD framework has been applied for the studies of polarised $J/\psi$ production at different energies (Tevatron and LHC). Theoretical uncertainties (UGDFs, scale choice and high order corrections) of the QCD mechanism under consideration have been investigated in detail. As a crucial test of our calculations, results for unpolarised $J/\psi$ production have been compared to the previous calculations in the literature and to the experimental data on prompt $J/\psi$ production from radiative $\chi_c$ decays at Tevatron $\sqrt{s} = 1.8$ TeV. We confirmed that $k_\perp$ factorisation framework leads to good description of the experimental data without the color octet contribution at the leading order of perturbation theory. Additionally we make predictions for the differential cross sections (in meson transverse momentum and rapidity) for the unpolarised $\chi_c$ and $J/\psi$ (from $\chi_c$ decays) production at the LHC energies (7 TeV and 14 TeV).

- Off-shell hard matrix elements for the polarised $\chi_c$ production have been derived in covariant form as well as in the helicity frame. The angular distribution of unpolarised $J/\psi$ and muonic pair have been analytically derived and the polarisation parameters $\lambda_\theta$, $\lambda_\phi$ and $\lambda_{\theta\phi}$ are obtained. These polarisation characteristics are calculated for all incident charmonia states $\chi_c^{J=1}$, $\chi_c^{J=2}$, $\chi_c$ (total) and $J/\psi$ from $\chi_c$. They have been analysed numerically in four different polarisation frames (Helicity, Collin-Soper, Gottfried-Jackson and Target frames). Results demonstrate the vanishing polarisation of $J/\psi$ coming from $\chi_c$ decays at all energies, thus confirming previous hypothesis declared in the literature. We have shown that proposed approach based on $k_\perp$ factorisation in NRQCD framework leads to results compatible with small or negligible $J/\psi$ polarisation which is consistent with recent observations at the LHC.

- To complete this study of prompt $J/\psi$ polarisation observables we still need to evaluate the contributions from direct $J/\psi$ production and $J/\psi$ coming from $\psi'$ decays which is motivated by recent LHC data. To prove the self-consistency of the color singlet model we need to evaluate the octet contribution (coming from the s-channel gluon splitting) and to compare it to the singlet one not only for unpolarised cross sections but also for polarisation characteristics. We expect that its contribution to the $J/\psi$ polarisation will be much smaller than the one in the collinear NLO QCD approach, but it is important to quantify it.

After completion of this work these results will be published in a peer reviewed journal.


[33] P. Faccioli, Quarkonium 2010 Workshop Paris, How to measure $\chi_c$ and $\chi_b$ polarisations


[40] CALLIGARIS, Master thesis, Analysis of $J/\psi$ production with the CMS detector at the Large Hadron Collider

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[45] Abe et al., CDF Collaboration, FERMILAB-Conf-95/226-E, Production of $J/\psi$ Mesons From $\chi_c$ Meson Decays in pp Collisions at $s = 1.8$ TeV


